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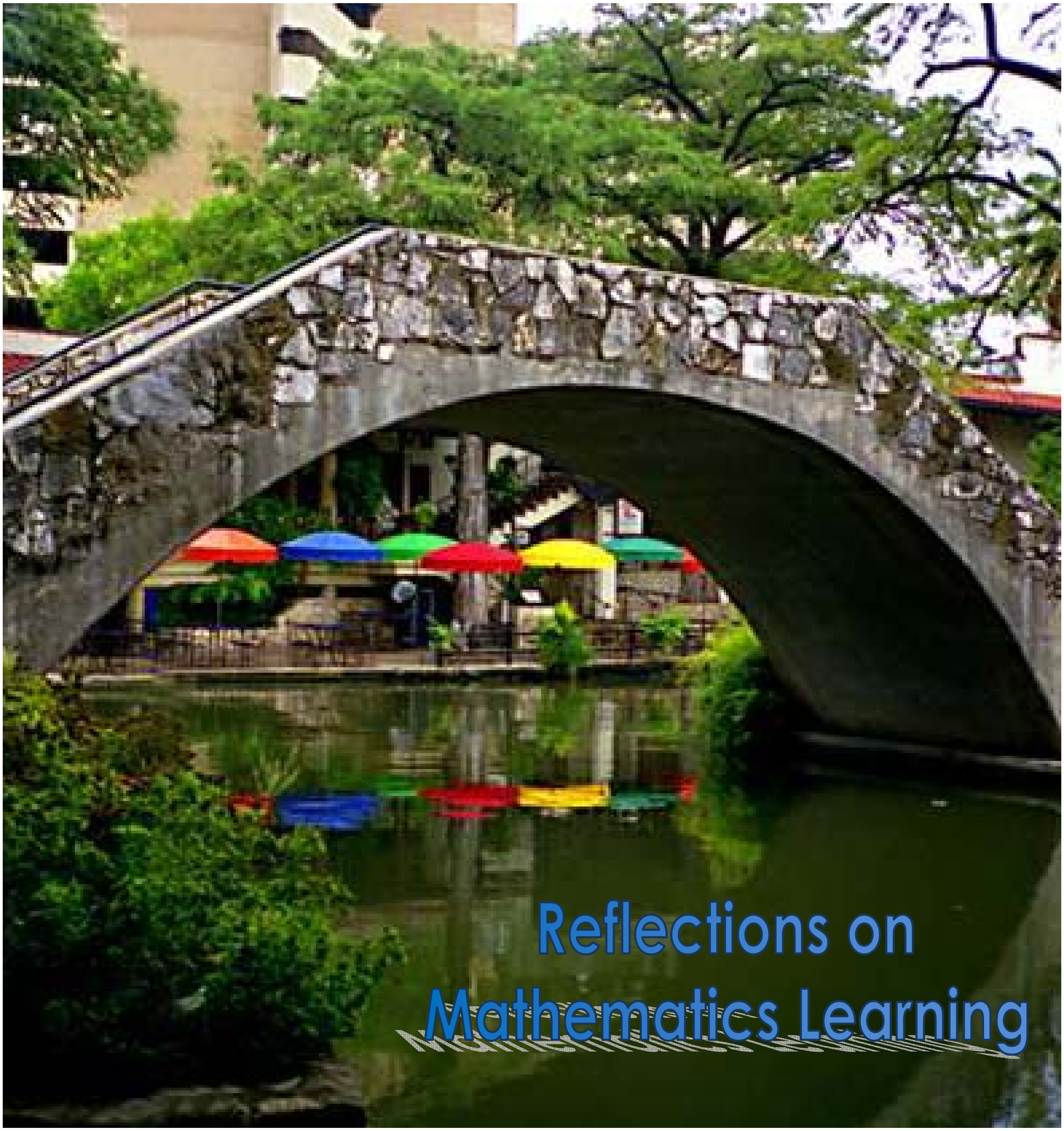

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RCML History

The Research Council on Mathematics Learning, formerly The Research Council for Diagnostic and Prescriptive Mathematics, grew from a seed planted at a 1974 national conference held at Kent State University. A need for an informational sharing structure in diagnostic, prescriptive, and remedial mathematics was identified by James W. Heddens. A group of invited professional educators convened to explore, discuss, and exchange ideas especially in regard to pupils having difficulty in learning mathematics. It was noted that there was considerable fragmentation and repetition of effort in research on learning deficiencies at all levels of student mathematical development. The discussions centered on how individuals could pool their talents, resources, and research efforts to help develop a body of knowledge. The intent was for teams of researchers to work together in collaborative research focused on solving student difficulties encountered in learning mathematics.

Specific areas identified were:

1. Synthesize innovative approaches.
2. Create insightful diagnostic instruments.
3. Create diagnostic techniques.
4. Develop new and interesting materials.
5. Examine research reporting strategies.

As a professional organization, the **Research Council on Mathematics Learning (RCML)** may be thought of as a vehicle to be used by its membership to accomplish specific goals. There is opportunity for everyone to actively participate in **RCML**. Indeed, such participation is mandatory if **RCML** is to continue to provide a forum for exploration, examination, and professional growth for mathematics educators at all levels.

The Founding Members of the Council are those individuals that presented papers at one of the first three National Remedial Mathematics Conferences held at Kent State University in 1974, 1975, and 1976.

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MATHEMATICS CONTENT KNOWLEDGE, ANXIETY, AND EFFICACY AMONG TRADITIONAL AND ALTERNATIVE CERTIFICATION ELEMENTARY SCHOOL TEACHERS

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The purpose of this study was to understand mathematical content knowledge, anxiety, and efficacy for mathematics elementary school traditional and alternative certification preservice and in-service teachers. The teachers in this study were given mathematics content examinations and mathematics anxiety and efficacy questionnaires in reform-based mathematics methods. Additionally, teachers were required to keep reflective teaching and learning journals throughout the semester. It was found that there were increases in content knowledge and efficacy, but not anxiety levels, over the course of the semester. There were no differences between traditional and alternative certification teachers in content knowledge, anxiety, and efficacy.

Mathematical content knowledge, anxiety, and teacher efficacy are important measures of teacher quality. Teacher content knowledge is important because it is directly related to student achievement (Hill, Rowan, & Ball, 2005). Mathematics anxiety is defined by Richardson and Suinn (1972) as the feeling of "tension and anxiety that interfere[s] with the manipulation of numbers and the solving of mathematical problems in a wide variety of life and academic settings" (Suinn & Winston, 2003, p. 167). Reducing teacher mathematics anxiety is critical for implementing student-centered reform-based teaching (Swars, Daane, & Giesen, 2006). Efficacy is a teacher's belief in his or her ability to teach effectively and positively affect student learning outcomes (Bandura, 1986; Enochs, Smith, & Huinker, 2000), and it is an important component for successful teaching.

Mathematics anxiety can cause a reduction in self-confidence (Rossan, 2006; Tobias, 1993), and teachers with mathematics anxiety tend to teach using traditional methods, such as lecture, instead of reform-based methods such as authentic problem solving (Swars et al., 2006). High mathematics anxiety can lead to avoidance of teaching higher level mathematics and negative attitudes can be transferred to students (Swars et al., 2006; Trice & Ogden, 1986). When students have lower level knowledge of mathematics and anxiety, they may avoid higher-level mathematics courses which can hinder career opportunities (Ma, 1999; Tobias, 1993).

Few studies have addressed mathematics knowledge with teacher efficacy (Newton, Leonard, Evans, & Eastburn, 2012; Swars et al., 2006). Newton et al. (2012) found a relationship between mathematics content knowledge and concepts of efficacy for elementary teachers taking a mathematics methods course. Further, Swars et al. (2006) found lower mathematics anxiety was related to higher concepts of efficacy, and found an increase in teacher efficacy over the course of an elementary mathematics methods class. Additionally, it is possible that beliefs about efficacy may be a greater variable in quality teaching than content knowledge alone (Bandura, 1986; Ernest, 1989).

Theoretical Framework

Ball, Hill, and Bass (2005) emphasized the importance of strong content knowledge for elementary school teachers and found that teachers who teach students of low socioeconomic status were less likely to have stronger content knowledge than teachers who did not teach these types of students, and teachers with stronger content knowledge had higher achieving students. Additionally, the gains in achievement for students of higher content knowledge teachers were similar to the differences between students of different socioeconomic status.

Ma (1999) found an inverse relationship between mathematics anxiety and achievement in mathematics. One theoretical explanatory model contends that anxiety interferes with the recall of prior mathematical knowledge and thus hinders the person from performing well (Ma, 1999). Another model explains the anxiety in terms of previous poor performance (Tobias, 1985). For teachers, high mathematics anxiety can lead to avoidance of teaching higher level mathematics and negative attitudes can be transferred to students (Swars et al., 2006; Trice & Ogden, 1986).

Bandura (1986) found that teacher efficacy can be subdivided into a teacher’s belief in his or her ability to teach effectively, and his or her belief in affecting student learning outcomes despite external factors. Teachers who feel that they cannot effectively teach mathematics and affect student learning are more likely to avoid teaching from an inquiry and student-centered approach for conceptual understanding (Swars et al., 2006).

Research Questions

1. What differences existed between teachers’ mathematical content knowledge, anxiety, and efficacy before and after an elementary mathematics methods course?

2. Were there differences in mathematical content knowledge, anxiety, and efficacy between traditional and alternative certification teachers?
3. What were teachers' beliefs about teaching and learning mathematics?

Methodology

The methodology of this study involved both quantitative and qualitative methods. The sample in this study consisted of 65 preservice and in-service teachers in a traditional (N = 28) and alternative certification (N = 37) master's degree program. About 25% of the participants were male and about 75% of the participants were female. Participants were enrolled in three reform-based elementary mathematics methods sections, which involved both pedagogical and content instruction and emphasized learning through an inquiry approach, problem solving, and mathematics for understanding.

Teachers were given mathematics content examinations and anxiety and efficacy questionnaires at the beginning and the end of the semester. The mathematics content examination consisted of 20 multiple choice items that measured knowledge of number sense, fractions, decimals, and percents (6 items); probability and statistics (4 items); measurement and geometry (5 items); and algebra (5 items), and was based on the PRAXIS mathematics examination (Educational Testing Service, 2005), as adapted by Newton et al. (2012). In New York teachers do not take the PRAXIS, but rather a New York State specific examination, so using the PRAXIS was appropriate for this study. Possible scores ranged from zero to 20 points.

The mathematics anxiety questionnaire was adapted from the brief version of the Mathematics Anxiety Rating Scale (MARS) developed by Suinn and Winston (2003) based upon the original MARS created by Richardson and Suinn (1972). The original version had 98 items measuring mathematics anxiety, and the brief version had 30 items derived from the original version through the use of factor analysis to reduce the size of the instrument in order to reduce administration time. Suinn and Winston (2003) found the brief version had similar reliability and validity as the original version. Two additional items were added by the researcher of the present study to include algebra and geometry anxiety. The MARS used a five-point Likert scale with choices indicating the participant's level of mathematics anxiety for a given item. The choices, referring to level of anxiety, were: very much, much, a fair amount, a little, and none at all. A higher score on this instrument represented less mathematics anxiety.

The efficacy questionnaire was the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) developed by Enochs et al. (2000), and measured concepts of teacher efficacy. The MTEBI was a 21-item five-point Likert scale instrument with choices of strongly agree, agree, uncertain, disagree, and strongly disagree, and was grounded in the theoretical framework of Bandura's (1986) efficacy theory. The MTEBI contained two subscales: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) with 13 and 8 items, respectively. Possible scores ranged from 13 to 65 on the PMTE, and 8 to 40 on the MTOE. The PMTE specifically measured a teacher's concept of his or her ability to effectively teach mathematics. The MTOE specifically measured a teacher's belief in his or her ability to directly affect student learning outcomes. Enochs et al. (2000) found the PMTE and MTOE had Cronbach alpha coefficients of 0.88 and 0.77, respectively.

Teachers were required to keep reflective journals on their teaching and learning over the course of the semester, which provided qualitative data regarding their attitudes toward teaching and learning mathematics. The teaching and learning journals had four entries each, one for each month of the semester.

Results

Research question one was answered using paired samples *t*-tests (see Table 1). There were statistically significant increases found over the semester for the mathematics content test, PMTE, and MTOE. However, no statistically significant difference was found for MARS. The mathematics content test had a large effect size, while the efficacy measures had small to medium effect sizes.

Table 1

Paired Samples t-Test Results on Content, Anxiety, and Efficacy

Assessment	Mean	SD	*t*-value	*d*-value
Mathematics Content Pre-Test	68.08	18.555	-9.096**	1.081
Mathematics Content Post-Test	86.08	14.510		
MARS Pre-Test	3.43	0.752	-0.758	
MARS Post-Test	3.49	0.611		
PMTE Pre-Test	3.51	0.615	-3.062**	0.382

PMTE Post-Test	3.72	0.479		
MTOE Pre-Test	3.52	0.484	-3.351**	0.460
MTOE Post-Test	3.74	0.473		

Note. $N = 65$, $df = 64$, two-tailed
** $p < 0.01$

Research question two was answered using independent samples *t*-tests (see Table 2). There were no statistically significant differences found between traditional students and alternative certification students. The closest variable to statistical significance was a difference between the traditional and alternative certification teachers on the MTOE post-test. The *p*-value was 0.083, and hence significant only at the 0.10 level. Of all variables it was expected that there would be a difference here since the alternative certification teachers have already encountered the realities of the classroom and should have lower MTOE scores. It is possible that there was no significant difference due to the strong field experience that the traditional students experience. The university in which this study took place places a very strong emphasis on field experiences for preservice teachers, and each 3-credit course in the graduate program requires 10 hours of fieldwork, which means preservice teachers experience at least 100 hours of field experience before student teaching.

Table 2
Independent Samples t-Test Results on Content, Anxiety, and Efficacy

Assessment	Mean	SD	*t*-value
Mathematics Content Pre-Test			
Traditional	64.64	16.212	-1.305
Alternative	70.68	19.971	
Mathematics Content Post-Test			
Traditional	85.54	15.416	-0.206
Alternative	86.49	13.987	
MARS Pre-Test			
Traditional	3.31	0.729	-1.040

Alternative	3.51	0.768	
MARS Post-Test			
Traditional	3.47	0.644	-0.174
Alternative	3.50	0.593	
PMTE Pre-Test			
Traditional	3.60	0.597	1.045
Alternative	3.44	0.627	
PMTE Post-Test			
Traditional	3.77	0.504	0.633
Alternative	3.69	0.463	
MTOE Pre-Test			
Traditional	3.53	0.521	0.228
Alternative	3.50	0.462	
MTOE Post-Test			
Traditional	3.86	0.460	1.759
Alternative	3.65	0.470	

Note. N = 65, df = 63, two-tailed

Research question three was answered using teacher reflections. Teachers were required to reflect upon their own teaching and learning in the methods course in their reflective journals. The teaching journals revealed teachers found the lack of student conceptual understanding to be one of the biggest issues for them. Many teachers discussed the challenges and rewards of working with special education populations, which is not surprising considering that many of the teachers in this study were currently teaching students with special needs or were interested in teaching students with special needs. Some teachers mentioned classroom management problems, but not as many teachers expressed this concern as would be expected. It should be noted, however, that one teacher did report she was attacked by a student and had to go to the hospital. Another teacher reported she had to file legal charges against a student who sexually harassed her and followed her to the train station. Fortunately, she reported that her experiences had improved over the semester. Fortunately, these were isolated incidents.

The learning journals revealed that the teachers found the course's emphasis on problem solving, conceptual understanding, real-world connections, and teaching using technology and manipulatives to be most helpful. It should be noted that many teachers wrote about an article that was presented to the class on virtual manipulatives, which focused on the various websites that offer virtual use of manipulatives. To a lesser extent teachers found the emphasis on motivational techniques, microteaching, differentiation, and literacy in mathematics to be helpful. Overwhelmingly teachers had expressed their own apprehension about the course in the beginning of the semester and wrote extensively about their own mathematics anxiety, possibly motivated by the MARS instrument at the beginning of the semester. Many teachers cited their own lack of conceptual understanding to be a cause of their own mathematics anxiety. Over the semester many teachers expressed a gain in confidence in teaching mathematics due to increased conceptual understanding due to the manner in which the course was conducted, but this was not reflected in their MARS post-test scores. Many teachers expressed concerns about teaching the *Everyday Mathematics* (University of Chicago School Mathematics Project, 2007) curriculum. To alleviate some of this anxiety the course focused on *Everyday Mathematics* content with an emphasis on understanding the various *Everyday Mathematics* techniques. Subsequently, many teachers focused on their growing confidence in using the *Everyday Mathematics* curriculum. One teacher wrote on the course's focus on conceptual understanding of fractions, which she claimed helped her greatly and will assist her in her own teaching of fractions.

Discussion

It was found that over the course of a graduate level reform-based mathematics methods course preservice and in-service teachers had an increase in mathematical content knowledge and efficacy, both in terms of personal concepts of effective teaching and student outcome expectancy. Many teachers indicated a decrease in mathematics anxiety and a gain in their own mathematical confidence in their reflective journals, but it was surprising that results from the MARS instrument did not indicate this.

No differences were found between traditional and alternative certification teachers for mathematics content knowledge, anxiety, or efficacy. However, there was a difference in student outcome expectancy on the post-test between traditional and alternative certification teachers at the 0.10 level. It was expected that traditional preservice teachers would have higher outcomes expectancy than alternative certification teachers, due to the realities of the classroom. However, recall that the traditional preservice teachers are required to have 10 hours of fieldwork for each 3-credit course in the graduate program. It is hypothesized that if the traditional preservice teachers did not have the strong fieldwork component in their program, there would have been a greater difference in outcome expectancy.

Findings in this study indicated that a reform-based methods course, coupled with field experiences, can improve teacher mathematical content knowledge and efficacy. However, further study is needed to determine the effects of coursework and fieldwork on teacher mathematics anxiety. The results of this study indicated there were many similarities between traditionally and alternatively prepared teachers. In order to ensure that the students are receiving the best possible educations, further studies on teacher quality and student success are needed.

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RIGOROUS MATH COURSES FOR MIDDLE-SCHOOL MATH TEACHERS

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We describe our five-year professional development project targeting middle-school math teachers. A primary focus of the project is providing the teachers with a deep conceptual understanding of the mathematics taught in the middle grades. Our analyses to date indicate a positive impact on the teachers' conceptual knowledge, their math knowledge for teaching, and their self-efficacy for teaching mathematics to diverse student populations. Preliminary findings also suggest a positive impact on teachers' classroom practices.

It is generally accepted that teachers of mathematics need to possess a deep conceptual understanding of the elementary mathematics they are teaching (Ma, 1999; CBMS, 2012) and need to have special knowledge related to the teaching of mathematics, often referred to as mathematics knowledge for teaching or MKT (Shulman, 1987; Ball et al, 2008). In addition, effective teachers need a strong belief in their ability to teach mathematics to diverse student populations (Hernandez et al., 2008; Moschkovich, 2002). In this paper we describe a five-year professional development project designed to increase the conceptual understanding, the MKT, and the math teaching self-efficacy of in-service middle-school math teachers in a large rural area of West Texas. We then provide a survey of the results we have obtained in relation to these teacher outcomes. Finally, we describe our continuing efforts to further document the long-term project effects on not only these teacher outcomes but also on teachers' classroom practice.

Project Description

The West Texas Middle School Math Partnership, WTMSMP, is a five-year professional development (PD) project funded by the National Science Foundation Math Science Partnership program, beginning January 1, 2009 and ending December 31, 2013. During this time period, two cohorts of middle-school math teachers completed a sequence of three graduate level summer math courses: Cohort 1 (n=65) in 2009 (Course 1), 2010 (Course 2), and 2011 (Course 3); Cohort 2 (n=84) in 2011 (Course 1), 2012 (Course 2), and 2013 (Course 3). These courses were taught at four partnering institutions of higher education, IHE, located fairly uniformly throughout a very large region of West Texas. The courses were taught by math faculty from each respective IHE, assisted by designated WTMSMP math faculty. All instructors of record had PhDs in mathematics. The focus of this paper is on the outcomes associated with teachers' completion of this mathematically intense PD experience.

Project Courses

Each course consisted of 48 contact hours taking place on the campus of each partner IHE during a two week period (three hours each Monday afternoon; six hours each Tuesday, Wednesday, Thursday; and three hours each Friday morning). Each course targeted a particular area of mathematics deemed critical for teachers of middle-school math: algebra concepts for Course 1; geometry concepts for Course 2; and probability and statistics concepts for Course 3. Each course also included a half-day workshop on an ancillary topic: Math Self-efficacy in Course 1; English Language Learners in Course 2; and Cultural Diversity in Course 3.

The textbooks for courses 1 and 2 were written by the WTMSMP Principal Investigator (PI), a mathematician with a PhD in pure math and lead author on this paper. The third course textbook was co-authored by the project PI and two other WTMSMP math faculty, one with a PhD in applied math and the other with an MS in statistics. Each textbook begins by identifying the fundamental concept in its area and then proceeds to develop the area in a mathematically logical progression. It is important to note that none of the three textbooks were aligned in any way with any preset curriculum standards or assessment measures.

The Course 1 text (*Integers and Fractions: An Investigation into the Algebraic Structure of Our Numbers*) begins with the definition of a positive integer from the view point of Bertrand Russell (Russell, 1956). The group and ring structures of the integers are derived, followed by the field structure of the rational numbers. Representations of the rational numbers, fractions and decimals, are covered in detail including the meaning of an infinitely repeating decimal "representation" of a fraction. Finally the Least Upper Bound Principle is invoked to produce the existence of irrational numbers (Harris, 2009).

The Course 2 text (*Measures of Size in 0, 1, 2, and 3 Dimensions*) begins with the size of a set of discrete points (size in 0 dimensions) being defined as the number of points in the set. A fundamental object in each subsequent higher dimension is defined via a vertical translation of the fundamental object in one lower dimension, and its size is defined to be the product of the size of the lower dimensional object times the vertical translation distance. All the usual formulas for size of polygonal objects are derived from the sizes of the fundamental objects. The circumference of a circle stems from the definition of the number π, the area of a circle is defined in terms of the limit of the areas of regular polygons, and the volume of a sphere is gotten from the classical argument of Archimedes (Archimedes, 1912). The text ends with a discussion of fractals and fractional dimensions (Harris, 2010).

The Course 3 text (*Concepts of Probability and Statistics*) begins with the basic concept of discrete theoretical probability; namely, the probability that a point chosen randomly from a finite set of points S (sample space) is in a subset E (event) of S is the ratio $\frac{Size(E)}{Size(S)}$. After counting techniques, the concepts of random variables, probability distributions, and expected values are introduced. These concepts are then extended to continuous probability involving samples spaces in 1, 2, and 3 dimensions, leading to the standard normal density distribution. The text ends with a discussion of statistical terms and concepts often encountered in the middle grade classroom and state assessments (Harris, et al, 2011).

Project Participants

Sixty-one teachers from Cohort 1 and 85 teachers from Cohort 2 completed all three courses. In each cohort approximately 80% were women. Overall the participants reported having 0 to 40 years of experience teaching math (average a little over 9 for each cohort). In terms of mathematics background, teachers reported having taken as few as 0 college level math courses to as many as 8 (average approximately 3.5 for each cohort). Points to be stressed are

1) There is no significant difference between cohorts 1 and 2 in any of the categories reported above. (This is not surprising since they are all from the same region of Texas.)
2) There is great disparity among participants' experience teaching math.
3) There is great disparity among the undergraduate math backgrounds of the participants.

Project Results

In order to make changes if necessary, it was important to assess the project's impact on Cohort 1 during the first two years (2009 and 2010) prior to the beginning of the second Cohort 2's experience with the project in the third year (2011). Impact was investigated by determining the extent to which Cohort 1's project participation was associated with increases in key programs outcomes: mathematics conceptual knowledge (MCK), MKT, and self-efficacy for teaching mathematics?

Using widely accepted and validated MKT scales (Schilling, et al, 2007; Hill, et al, 2007;Schilling, 2007), the WTMSMP researchers were able to assess the project impact on Cohort 1 teachers' knowledge for teaching Number Concepts and Operations, Algebra, and Geometry (Harris, et al, 2011). In addition, the researchers used a locally created instrument (yet to be fully validated) to assess the impact on the teachers' MCK for geometry. All measures increased from pre-year 1 to post-year 2 with the increases in MKT for geometry and MCK being statistically significant. Since there was no alignment between the course texts and the MKT measures, the researchers were pleased to see growth in the MKT scales. There was no indication that years of experience had any relationship to participants' MKT growth. The math background of the participants did affect the MKT and MCK scores with those who had taken courses beyond college algebra scoring, on average, higher on all measures, at all instrument administrations (pre- and post-course delivery for each of the previous two summers for a total of four time points). However, the gains were parallel meaning that the value added, as a portion of initial knowledge, was the same for both groups.

The researchers believe this is a significant finding. Regardless of initial background, the rate of growth was the same for all participants. This finding is particularly interesting in light of the construct measured by the MKT measure. The MKT is not a direct measure of mathematics concepts. That is, it captures mathematics content in the context of teaching scenarios and decision making (Schilling, et al, 2007; Hill, et al, 2007;Schilling, 2007). Therefore, this finding suggests that a focus on pure mathematics can yield important changes to teachers MKT, provided the experience allows teachers' an opportunity to translate the material into meaningful classroom situations. This course feature was carefully integrated into each course. The results described in the continuing work section are consistent with this interpretation of this finding.

Using well established measures, the WTMSMP researchers also found evidence supporting participants' growth in teaching self-efficacy associated with course completion (Stevens et al., 2013a). As expected, participants reported increasingly higher levels of confidence in their ability to provide instruction and engage students at each of the four time steps. Factoring in teachers' levels of math background revealed participants who took fewer college math courses reported higher initial levels of teaching efficacy. This finding could be due to these teachers having taken more pedagogically focused courses in lieu of mathematics, although the data were not available to explore this explanation. Regardless of math background, the teaching self-efficacy of all teachers increased, with the teaching efficacy scores of those with more math background approaching the scores of those with less math background by the end of Course 2 (Stevens et al., 2013a).

At Cohort 1's completion of WTMSMP coursework, the shape of participants' MKT growth over the three years of the project was assessed and revealed statistically significant linear growth for Algebra and Number Concepts (Stevens et al., 2013b). The teachers' growth on the Geometry MKT measure, however, was nonlinear with a large increase after Course 2 and little additional growth by the end of Course 3. Algebra and Number Concepts scores grew consistently over the three years of the project despite variation in course content. This growth is consistent with Ma's observation (Ma, 1999) that teachers' conceptual understanding of mathematics develops as they teach and interact with their students. The fact that geometry MKT scores did not grow during Course 3 could be attributed to the closer alignment of the geometry MKT measures and the content of Course 2.

Documenting Cohort 1 teachers' growth across their WTMSMP involvement was important; however, it was also important to show that participants' final MKT scores exceeded those of similar teachers who had no experience with the project. With Cohort 2 starting at the end of Cohort 1's participation, WTMWMP researchers were able to compare Cohort 1's final scores to the pretests of Cohort 2. Because the two groups of teachers reported similar levels of math background and experience teaching prior to the start of participation, this comparison was appropriate. Results indicated that although the MKT scores of Cohort 1 and Cohort 2 did not differ at the pretest, Cohort 1's final MKT scores were significantly higher than Cohort 2's pretest scores. Thus, after WTMSMP completion, participants outperformed similar teachers on measures of MKT (Stevens et al, 2013b).

Continuing Work

In addition to documenting Cohort 2 participants' growth in MKT and teaching efficacy, continuing work will also focus on what aspects of the WTMSMP project were influential in supporting all participants' growth, as well as what changes in teacher practices can be observed in middle school classrooms. Understanding what project participants perceived as most beneficial to their learning will be evaluated through analysis of the teachers' Q-sorts. "The objective in Q-methodology is to describe typical representations of different viewpoints rather than to find the proportion of individuals with specific viewpoints" (Akhtar-Danesh, Baumann,

& Cordingly, 2008). All participants were asked to sort and rank-order the aspects of the WTMSMP courses that most benefited their learning. We will use factor analysis to organize participants' perceptions into categories. This method was successfully piloted by the researchers (Stevens et al., 2009). The analysis of the Q-sort data for Cohort 1 upon completion of Course 1 revealed three approaches to learning; participants who focused on gaining competence, participants who preferred to be in control of their learning, and participants who benefited most from social learning (Stevens et al., 2013c). Cohort 1 teachers appeared to interact with Course 1 strategies and activities in different manners determined by their divergent approaches to learning. These results underscore the need to incorporate variety in course strategies and activities, and suggest the need for long term, intensive professional development activities.

To understand how participants are taking their knowledge into their classrooms, the researchers developed the Students Perceptions of Teacher Successes (SPoTS) instrument, which allows public school students the opportunity to quickly rate their teachers on key behaviors promoted by the WTMSMP project. Initial evaluation of the SPoTS yielded positive support for its usefulness in understanding teacher practices (Stevens et al., 2013d). This instrument will be used to investigate participants' ongoing use of WTMSMP content in their classrooms.

Discussion

The course structures were driven by the logical development of the math content resulting in no direct alignment with the MKT measures or the self-efficacy measures. And yet upon completion of the three courses participants had significant increases in their mathematical knowledge for teaching, math self-efficacy, and self-efficacy for teaching math. We believe there are multiple factors that contributed to these increases.

First and foremost, each course provided a deep theoretical development of the mathematics taught in the middle grades: the algebraic structure of the rational number field in Course 1; the fundamental concept of size in 0, 1, 2, and 3 dimensions in Course 2; the transition from discrete to continuous probability in Course 3. In addition the courses were taught by research mathematicians, each with a PhD in mathematics and a passion for, and appreciation of the beauty of, the mathematics being studied. Based on participant comments gathered at the final WTMSMP retreat we believe the passion and appreciation expressed by professional mathematicians for the math encountered in their own middle-school classrooms was a source of inspiration for many of the project participants.

Another factor was the emersion effect. Each course consisted of 16 three-hour sessions completed over a two-week summer period. In each session the teachers were exposed to a theoretical development of the concepts and then worked in groups to create concrete models or demonstrations of the concepts suitable for use in their own classrooms. This produced a collaborative atmosphere in which all participants, the teachers and the IHE mathematicians, interacted as colleagues. This interaction may have contributed to a change in teachers' conceptions of mathematics and thus measurable changes in our measures.

Of the 149 teachers who began the program 132 (59 from Cohort 1 and 73 from Cohort 2) successfully completed all three courses. The majority of those who dropped did so because of changes in jobs. We believe these results demonstrate that this PD program can be of significant benefit for middle school math teachers with wide variation in teaching experience and in undergraduate mathematics backgrounds. Moreover, we believe the kind of results we are seeing are not likely attainable using the traditional professional development model: half-day, whole-day, or weekend workshops scattered throughout the year. We believe that the intense, long-term, rigorous focus on the specific mathematics content taught in the middle grades was critical to the success of our program.

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PRE-SERVICE TEACHERS AND THE REPRESENTATIVENESS HEURISTIC

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In this mixed methods study, sixty-six pre-service teachers were surveyed and interviewed concerning the presence of the representativeness heuristic and reasoning while solving probability tasks. Two tendencies were observed; the first was a narrow focus on a 50/50. The second was an 'anything can happen' view. Additionally, there was a difference in results based on the problems' context. The implications this has for teacher educators and teacher education programs are discussed.

On a daily basis, people encounter statistical and probabilistic information regarding commerce, finance, lifestyle, health, politics, and community participation. The prevalence of this type of information present in modern society requires the knowledge, understanding, and critical evaluation of data (Shaughnessy, 2007; Gal, 2002). Shaughnessy (2007) referred to this as statistical literacy, which he defined as the understanding and ability to analyze and apply statistical and probabilistic information in an appropriate and confident manner. Therefore, a general education curriculum should include statistical literacy to prepare students to meet society's increased demand for the understanding and evaluation of information (NCTM, 2000, Shaughnessy, 2007). Concerning the instruction and learning, problem solving of this nature requires an understanding of the mathematics applied, as well as the reasoning strategies and processes specific to statistics and probability (Cobb & Moore, 1997; Pfannkuch & Wild, 2004). For this to be possible, teacher competence is essential (Shaughnessy, 2007).

Earlier seminal studies have demonstrated that students use erroneous strategies including the outcome approach and the representativeness and availability heuristics (Kahneman & Tversky, 1972; Konold et al., 1993; Green, 1983). Research involving misconceptions has increased the knowledge base by advancing the understanding of students' learning and reasoning. While many studies have involved K-12 and college students (Green, 1983; Konold, 1989; Shaughnessy, 2007), information regarding pre-service teachers' use of these heuristics is limited. Shaughnessy (2007) emphasized the emerging need for research into teachers' understanding of conceptions and beliefs in the area of statistics and probability. This study will help answer this call by examining how elementary pre-service teachers reason probabilistically, and whether they, too, use the representativeness heuristic.

Literature Review

Seminal research in the area of the probabilistic misconceptions and heuristics found that the associations people make influence their approaches to probabilistic problem solving (Kahneman, Slovic & Tversky, 1982; Konold et al., 1993). In the representativeness heuristic, a heuristic proposed and studied in the previously mentioned seminal research, a person determines an outcome based on how it represents the sample or the selection process while neglecting relevant probability concepts. For instance, after six flips of a fair coin with all resulting in heads, a person may predict tails for the following flip in order to approach an expected one-to-one distribution of heads and tails. Alternately, a person might focus on the distribution percentage in samples while neglecting the sample size. Konold et al. (1993) contend that task presentation and wording affect reasoning and the tendency to apply this heuristic. Regarding heuristics and problem solving, the seminal research and observations of their prevalent use through all levels of learning and education demonstrate their role in probabilistic reasoning, and emphasize their importance in understanding student perceptions and thinking. The Common Core (Council of Chief State School Officers [CCSSO], 2010) is being widely implemented across the United States and with its increased emphasis on probability and statistics at the middle grades and secondary levels there is once again a need to revisit this knowledge base and to prepare future teachers in this essential area. Pre-service teachers require a solid foundation for instructing students in probability and statistics, which requires that they understand some of the possible misconceptions they may be faced with in the classroom. One of the first steps in helping them understand the heuristics that their future students might display is to see if these heuristics still persist into post-secondary education and whether they are present in the problem solving processes of future teachers. Thus, this study concentrated on the following related research questions: 1. Does a question's context influence how pre-service teachers reason probabilistically and in similar situations? and 2. In what ways do pre-service teachers use the representativeness heuristic?

Methodology

This research was conducted at a mid-sized, four-year, public university in the Midwest that has teaching as a primary focus. The 66 pre-service teachers were enrolled in various teacher education classes designed to prepare them to teach mathematics in elementary and middle school. Consenting participants completed a set of four binomial probability tasks, two that represented sequences and two that were predictive. Of the original 66 participants, nine were purposefully selected for follow-up interviews. The nine interview participants were chosen based on a range of task success, and a variety of written responses and their associated levels of reasoning, which included both subjective and quantitative reasoning and references. Survey questions were drawn and adapted from various sources and formatted with multiple-choice responses and a space for an explanation for the selected answer (Shaughnessy, 2003; Green, 1983; Konold et al., 1993). Question 1 (Q1) was a sequence task and involved a container of 10 white (W) and 10 black (B) marbles and asked for the most likely sequence descriptive of seven draws, with replacement: a) BWWBWBW, b) BWWWBWB, c) WWWWWWW, d) WBBBWWW, and e) All four sequences are equally likely. Question 2 (Q2) was predictive and asked for the most likely result of a seventh flip of a fair coin after six flips resulted in all heads with choices: a) A Head, b) A Tail, and c) …equally likely. Question 3 was another sequence question and asked for the least likely sequence of boys and girls in a family of six children with the five choices that were analogous to Q1. Question 4 (Q4) paralleled Q3 as predictive with a box of 20 red and 20 green balls and an all green result of the six previous draws with replacement. Nine participants were selected for follow-up interviews, which were audio recorded and transcribed for accuracy. In these interviews, participants were asked to express their thinking as they answered the tasks and were given two additional problems that were conceptually related to the previous tasks.

We used a theoretical framework developed by Jones et al. (1997) to code our responses. This framework includes four separate aspects of statistical reasoning, but for this study, we only utilized the portion that addressed probabilistic understanding. Table 1 summarizes the relevant portion of this framework.

Table 1
Probability Framework Jones et al. (1997)

Probability of an Event			
Level 1 Subjective	Level 2 Transitional	Level 3 Informal Quantitative	Level 4 Numerical
1. Predicts most/least likely	1. Predicts most/least likely event based on	1. Predicts most/least likely event based on	1. Predicts most/least likely events for single

event based on subjective judgments	quantitative judgments but may revert to subjective judgments	quantitative judgments 2. Uses numbers informally to compare probabilities	stage experiments 2. Assigns a numerical probability to an event

Findings

To answer our first research question regarding the difference in context of the questions, we used a one-way ANOVA with a post hoc test as well as simple descriptive statistics. Initially, we looked at the correctly answered percentages for each question and found: 67% answered question 1 correctly, 91% answered question 2 correctly, 47% answered question 3 correctly, and 82% answered question 4 correctly. When the surveys were quantitatively analyzed, we found that the mean response level for all four questions was statistically different, with an observed p-value less than .001. This suggests that at least one of the four means was different from the others. To determine where this difference occurred we used Tukey's HSD and found that the means for questions 1 and 3 were the same and the means for both questions 2 and 4 were different from each other and from the mean for questions 1 and 3.

Table 2
Results from One-Way ANOVA and Tukey's HSD

Question & Type	Mean	Equal Means	Equal Means	Equal Means
Q #1 - Sequence/Most	1.8182	X		
Q #2 - Predictive/Most	2.6212		X	
Q #3 - Sequence/Least	1.7879	X		
Q #4 - Predictive/Least	2.2576			X

Thus, there is some contextual difference in the types of reasoning used by participants. Generally speaking, participants were more successful in answering questions 2 and 4, which focused on predicting the next occurrence following a string of identical results, like choosing heads or tails after flipping six heads in a row. Errors were significantly higher with questions 1 and 3, while the statistical results showed a difference in reasoning level. These tasks focused on determining the most or least likelihood, respectively, of a given sequence occurring, like representation of birth order in a family of six children. To answer the second research question we analyzed each of the follow-up interviews qualitatively using grounded theory (Glaser & Strauss, 1967) to identify emerging trends in the data. While surveying to determine the presence and reasoning of the representativeness heuristic, two trends emerged during analysis. In the first trend, participants regularly demonstrated a focus on the binomial, or 50/50 chance as phrased by many. This narrow focus often caused them to ignore other concepts such as sample space. This suggests the presence of the representativeness heuristic where the focus on a 50/50 chance, caused participants to either chose or deny an even representation of the two options, like heads and tails, in certain responses. The second emerging trend was phrased as 'anything can happen', which denied the mathematical portion of the probability task. While not directly exhibiting the heuristic, the second focus showed an uncertainty that resulted in the correct answer without any associated understanding. Most students surveyed exhibited probabilistic uncertainty, which resulted in the application of either one or both of the observed trends.

In trend 1, we found that while recognizing the binomial aspect of the task, some used this narrow focus for applying the representativeness heuristic with an answer that represented the closest mix ratio of 50/50. During an interview, participant Ann reverted to the heuristic when asked about her least likely answer of all girls to question 3.

> Ann: Even though it is possible, it's just not quite as likely…I mean, I guess answer E is just as likely to happen, but then again [there is] a 50/50 drawing every single time. So, it's hard for me to see that it'll be the girl every time.

Of the two participants in the study that referred to sample space, Connor correctly referred to the 1/64 chance defined by the understanding of the sample space. He also mentioned replacement and independence of events. During the interview, Connor expressed an understanding of probability including the knowledge of a sequence as one possibility of the sample space and the equal likelihood of every given sequence. However, within a few minutes, when performing the additional tasks that were different from, yet similar to, the previous tasks, he reverted to the representativeness heuristic.

> Connor: Well by looking at the sequence, I mean, they are all, if it's in that particular order, they all have the same chances, but the least likely I think is going to be getting C, all peppermint.

This implies that the heuristic is fairly resistant to change even in light of realizations that would seem to contradict it.

Participants often referred to pieces of a sequence rather than the whole as with “each draw is 50/50, so there is a 50/50 chance of any of them” or “each birth is 50/50… [so] two in a row is more likely than three in a row.” By neglecting the sequence as part of the connected sample space, they focused on a 50/50 chance and the desire for a representative balance. Jennifer expressed this focus when responding, “If you knew that there was a 50/50 chance you could kind of run with that, which is what I did.”

The second trend saw the 50/50 chance in an opposite way. In this perspective, participants perceived the even probability, or a 50/50 chance as they phrased it, as an equal uncertainty about their answer, which led to the application of ‘anything can happen.’ Examples of this were observed in responses that ranged from “you just don’t know until it happens” or “they are all possible.” Further, this perception of ‘anything can happen’ often encouraged them to choose the correct response of “all are equally likely” even though this did not suggest that they understood the probabilistic ideas behind the task.

Finally, we used mixed methods (Creswell & Clark, 2011) to integrate the two analyses in such a way that they each helped illuminate each other. Regarding reasoning and success, the quantitative results suggest that there is a difference in context of the questions, and this conclusion is supported by the emerging trends discussed above. Thus, in response to our first research question, the quantitative results showed a marked difference in the accuracy depending on the context of the questions. The predictive questions exhibited clearer understanding than the most/least likely tasks. In response to the second research question, the most/least likely tasks promoted more subjective responses, which elicited the use of the representativeness heuristic and other misconceptions. Participants seemed narrowly focused on the idea of a 50/50 chance. As a result, they disregarded sample space and the sequence as a unit and resorted to using the representativeness heuristic. Of further concern, for some, the 50/50 perspective accentuated the uncertainty of the probability, which generated the ‘anything can happen’ concept while resulting with the correctly chosen answer of “all are equally likely.”

Discussion

Our observations highlight two areas of need in pre-service teacher training in the area of probability and statistics. The narrow focus of 50/50 shows the limited comprehension of the multiple layers of consideration required of probabilistic and statistical reasoning as with the inattention to sequences and sample space. Secondly, the 'anything can happen' is similar to the findings of Konold et al. (1993) and exhibits the continued conflict between the uncertainty aspect of probability and the comprehended, deterministic approach of mathematics.

Shaughnessy (2003) noted that many elementary and middle school pre-service teachers mentioned limited instruction, experience, and confidence, regarding their probability knowledge and abilities, which concurs with our findings. In addition, our analyses agree with Garfield and Ben-Zvi's (2007) suggestion that the ambiguity associated with underestimating and overestimating a student's understanding of basic probability concepts is an important concern in teacher education. Statistics and probability require a different way of thinking that considers the relationship between the mathematical numbers and the situational context (Cobb & Moore, 1997; Pfannkuch & Wild, 2004). Both of these aspects are necessary for teaching content and identifying misconceptions in themselves and future students. Therefore, the development of pedagogical and content knowledge in probability and statistics should include students' use of misconceptions (Garfield & Ben-Zvi, 2007). Many pre-service teachers have limited experience and even less pedagogical knowledge regarding statistics and probability. Thus, for the effective teaching of probability and statistics, as is necessary with the implementation of the Common Core (CCSSO, 2010), a broader and stronger understanding of the subject and associated misconceptions is called for. If pre-service teachers have not developed confidence and a solid content base in probability and statistics, leaving their knowledge limited and prone to misconceptions, there will continue to be a delay in the increase of student and adult statistical literacy.

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A MODEL FOR MATHEMATICS TEACHER PREPARATION

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A study was conducted to determine the factors that most effectively predict success in a mathematics teacher preparation program that involved a Summer Bridge and several research-based undergraduate experiences throughout the four years of the program. The single best predictor of college performance was unweighted high school grade point average. ACT scores and evaluations of applicant responses to questions and essays provided minimal additional predictive value. Long-term research can be conducted on prediction of retention in the program, job placement, and classroom teaching effectiveness.

The preparation of a mathematics teacher is a multi-faceted endeavor. The process needs to include an appropriate mix of mathematics content, learning psychology, pedagogy (including pedagogical content knowledge), and field experiences. While there is no single method to prepare a teacher, these components are typical of most teacher education programs. In the summer of 2009, Bowling Green State University in Ohio instituted a new program to enhance middle and high school teacher preparation in mathematics (and in science). Funding of student scholarships was secured through a grant and the university provided cost sharing to establish the Science and Math Education in ACTION program (ACTION). The goal was to accept approximately 25 students into a cohort each year, building the program to a maximum of about 100 students in four years. This competitive scholarship program required a process to most accurately select participants for a uniquely designed effective teacher preparation. We briefly describe the ACTION program here and explore what factors matter statistically when selecting suitable candidates. Since the strength of a teacher preparation program ultimately affects student learning, this study directly addresses the mission of the Research Council on Mathematics Learning in that it disseminates research "designed to understand and/or influence factors that affect mathematics learning"(2009, p. 1).

Landscape of Grant-Funded Teacher Education Programs

Components of grant-funded teacher education programs vary considerably, depending on the funding agency and the philosophy of the higher education institution. For example, a program at the University of Akron in Ohio involves a summer institute with a cohort model but does not feature first-year research projects or field-based internships in business and industry (Smith, 2013). A Woodrow Wilson Teaching Fellowship program in Cincinnati, Ohio, also uses a cohort model but requires that graduates teach in a high-need school upon completion of the program (University of Cincinnati, 2013).

Previous research supports the importance of teachers gaining experiences outside of the classroom to appreciate the application of mathematics in the working world, yet many teachers never get this opportunity. In fact, the real-world examples used by teachers who have not engaged in activities in business and industry tend to be restricted to what is mentioned in mathematics textbooks (Garri & Silverman, 2009). Furthermore, the use of summer bridge programs is becoming more common at higher education institutions throughout the U.S. These programs are typically designed to develop leadership, build community, and mentor incoming freshmen. However, most of these programs are intended for at-risk college students as opposed to challenging high-achieving undergraduates (Adams, 2012).

Context of ACTION

The ACTION program engages teacher candidates in the same coursework as non-ACTION students but adds several dimensions to the process. Prior to their first year of college, students participate in a residential, four-week, Summer Bridge Program to introduce them to the professors and content areas they will be studying for the next four years. Team building activities encourage students to bond with one another to form a supportive cohort of future educators. In the first year, undergraduates work in small groups with a faculty member from one of the sciences or mathematics, conducting research on a topic of the professor's choice. Students experience bench research as they hypothesize, collect and analyze data, and draw conclusions from the results. In the second year, undergraduates are placed in a community setting, such as a local business or agency, conducting an internship in which science and mathematics are used on the job every day. Students are required to develop lesson ideas that would help their classes to understand how science and mathematics are applied in the real world. Finally, teacher candidates are taught how to conduct classroom action research in the third year, during which time they research a topic and write a proposal for a classroom study. In their final year, during student teaching, undergraduates conduct an action research study, analyze, and write up their results in a final capstone paper. In the end, the intent is to provide teacher candidates with considerably more and deeper educational experiences than a standard teacher preparation program can provide.

An essential component in the ACTION program is teaching undergraduates how to conduct research in their own classroom. Benefits of teachers doing action research are numerous, including the generation of knowledge that can be applied to one's own classroom and promotion of reflection on practice (Hine, 2013). In addition, studies on the effects of having undergraduates conduct pure research in the sciences indicates that students learn how to think like a scientist and gain research as well as communication skills (Seymour, Hunter, Laursen, & Deantoni, 2004). In general, a review of the literature indicates that the model of providing a combination of pure research in science and mathematics, internships in business and industry, and conducting action research in a classroom is unique to the ACTION program.

Selection of participants in the ACTION program is based on a written application in which a high school senior must submit a transcript, ACT scores, a letter of recommendation, and responses to several essay questions designed to assess their ability to communicate and their desire to be a teacher. Beginning in 2013, top candidates were also interviewed online (using Skype or FaceTime) to assist with making final selection decisions. The research questions examined in this paper are the following: Of all of the attributes used for selection of ACTION scholars, which measures most accurately predicted success of the students such that success is measured by their undergraduate grade point average? Are there factors that did not significantly contribute to the prediction of success? How might the selection process be modified for improvement?

Just as the components of the programs vary by institution, the same is true for how institutions measure success of their students. While the ACTION program determines success as measured by undergraduate grade point average, a fellowship program at the University of Indianapolis uses the program's retention rate (U.S. News and World Report, 2013), the University of Akron measures success by its graduation rate (Korey, 2013), and others use job placement percentage (Indiana University, 2012). Other factors, such as involvement in student activities on campus, could be considered when measuring student success; however, no examples of this measure were identified in the literature review. Likewise, selection criteria for programs varies by institution, from high school performance to completion of essay questions, as are typical methods used by colleges for acceptance of students into any program. Thus, we aim to explore ACTION's approach to candidate acceptance.

Data Collection

In order to study the prediction of success of students, four cohorts from the ACTION program were selected for data analysis. A total of 109 participants were enrolled in the ACTION program after four academic years. Data were collected from students' files.

ACTION students' evaluation rating, combined ACT score, and high school unweighted grade point average (GPA) were treated as independent variables, and undergraduate GPA was the dependent variable. Students came from numerous high schools in Ohio; however, they participated in similar undergraduate teacher preparation courses (e.g., education and content-

Table 1
Descriptive Statistics for Undergraduate Students

Variables	Mean	SD	N
Undergraduate GPA	3.61	0.37	98
HS GPA	3.73	0.26	109
Undergraduate hours completed	80.66	40.57	98
Composite ACT	26.71	2.65	108
Evaluation Score	24.43	2.97	80

area coursework). Thus, statistical noise in undergraduate GPA was minimized. The evaluation scores consisted of ratings of essays and responses to questions submitted by ACTION students prior to their admittance into the program. For example, applicants were asked to "describe your role in one volunteer activity in the community that you chose to participate in (that was not required in any way)" and to "describe your interest in teaching science and/or mathematics." Applicants were scored on seven dimensions that included quality and quantity of high school activities, community involvement, and passion for teaching. Each dimension was scored on a scale of 1 to 5, for minimum and maximum scale scores of 7 and 35, respectively. Twenty-four of the 109 students did not have any ratings by an evaluator because the first year of the program did not involve the complex application process that was later adopted.

The evaluation rating was composed of three raters; however, only two raters scored each student's materials. Two raters were graduate university faculty and one was the program manager. One graduate university faculty member and the program manager rated each material.

Descriptive statistics of student-related factors are presented in Table 1, including the number of observations for each factor.

Data Analysis

Interrater reliability for the two reviewers' scores was explored using the Pearson correlation because the data were continuous. At first glance, the evaluators' scores showed little indication of differences in agreement (i.e., greater than two units). Interrater reliability greater than 0.80 is considered ideal (Lebreton, Burgess, Kaiser, Atchley, & James, 2003). Interrater reliability was examined using 80 students' evaluation ratings. The graduate faculty average rating was 24.16 (*SD* = 2.96) and the program director's average rating was 24.7 (*SD* = 2.93). The Pearson correlation coefficient was .84, which suggested ideal interrater reliability. This finding suggested that the reliability between raters was strong and that they were fairly calibrated in their ratings.

Ordinary least squares multiple regression was employed to examine whether any independent variable appropriately predicted the dependent variable, and if so to what degree? Pair-wise deletions were used when no data were available, which allowed the analyses to continue albeit the sample size changed. Model building examining the influence of the predictor variables on the outcomes was conducted by using backwards elimination, which sequentially deletes variables one at a time from the model that contains all of the variables (Agresti & Finlay, 2009). This procedure is best for creating a good set of predictors that explain a significant amount of variance in students' outcome (Agresti & Finlay, 2009). Furthermore, backwards elimination procedures support investigations for a parsimonious model and also attend to statistical and theoretical implications (Hamilton, 2009). The criterion to drop a variable was set at $p = .10$. The initial models including all predictors is shown below:

$$Undergraduate\_GPA = \alpha + \beta_1 HS\_GPA + \beta_2 Undergraduate\_Hours + \beta_3 ACT + \beta_4 Eval\_Score + \varepsilon$$

Covariates that were not significant at $p = .10$ were deleted and the model was re-estimated, which follows typical guidelines for model re-estimation (Agresti & Finlay, 2009). This process continued until only significant covariates remained in the model.

Results

All 109 participants' data were input into SPSS to explore the relationship between the independent and dependent variables. Backwards elimination indicated that the best model included one significant predictor: high school unweighted GPA. This model is shown below and results are provided in table 2:

$$Undergraduate\_GPA = \alpha + \beta_1 HS\_GPA + \varepsilon$$

Table 2

Undergraduate GPA predictors

Variables	Std. Coeff	Unstd. Coeff (SE)	t	Sig*	Conf. Int.
Constant	--	.74 (.61)	1.21	0.231	[-.48, 1.97]
HS GPA	0.48	.76 (.16)	4.63	0.0001	[.43, 1.09]

* two-tailed interpretations were used for these analyses.

This final model accounted for 27.7% of the variance in students' undergraduate GPA. This suggests that every one-point increase in high school GPA led to an approximate three-quarter point increase in undergraduate GPA.

Discussion and Conclusions

Analysis of the data indicates that the high school grade point average of applicants to the ACTION program is the single best indicator of performance in college coursework. Interestingly, none of the other measures – from ACT scores to recommendation letters and ratings on essays – significantly improved the predictability model for success. Simply put, the higher a student performed in high school, the higher grades the student was likely to earn while in college, regardless of what ACT or essays may also indicate. This is an important finding because it suggests that a simpler application process may be equally effective in selecting a cohort of accepted applicants, thus saving the applicants and university administrators time in preparing and reviewing materials.

When considering how to modify the application process, it is also important to consider the type of student that is desired in the program. For example, a stellar high school student may or may not possess the leadership skills and the desire to be a teacher that a more marginal student may possess (and vice versa). If program officials can identify students with the strongest desire to become teachers, acceptance of those students may affect the retention rate, both in the ACTION program and in a teaching career after college. In this case, "success" may be more accurately measured by considering retention rate in the teacher preparation program and in a teaching career at some point in the future. For the purposes of this study, we only consider the selection process in terms of predicting success in college courses.

Another area of research that is under investigation is to examine each component of the ACTION program and determine which of its features – Summer Bridge, freshman research, sophomore internships, junior/senior capstone projects – are the most beneficial in terms of most effectively preparing teachers. Through exit interviews of graduates and field-based interviews of ACTION alumni, data are being gathered to determine how to refine the program, and analyses will be conducted in the near future.

In many ways, the ACTION program is still in its infancy stages. In the first year, there was no formal application process other than submitting transcripts from high school. In the next three years, the process grew to include submission of recommendation letters and essays. By the fifth year, interviews had become part of the process. Additional data gathering and analysis, including long-term results of the program, need to be conducted to more adequately refine a selection process and a structure that will most effectively prepare teachers of mathematics for the classroom.

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DRAW YOURSELF LEARNING AND TEACHING MATHEMATICS: A COLLABORATIVE ANALYSIS

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To examine pre-service elementary teachers' enduring and dynamic aspects of disposition toward mathematics teaching and learning, participants were asked to draw themselves: 1) learning mathematics and 2) teaching mathematics, at the beginning and end of a mathematics methods course. Drawing methodology was employed because, "In a very real sense, drawings make parts of the self . . . visible" (Mitchell, Theron, Stuart, Smith, & Campbell, 2011, p. 19). Along with drawings, participants provided written explanations and participated in one-on-one interviews. This collaborative analysis technique has the potential for more authentic knowledge construction about teacher disposition.

This document will report initial findings of a larger study that employs drawing methodology as one of the sources for data collection and analysis. The completed study will also include a second phase of data collection and analysis of drawings as well as one-on-one interviews that will contribute to a deeper and more accurate identification of the meanings communicated through the drawings. This initial analysis, drawn from a randomly selected subset of participants, suggests further investigation into three potential conclusions: 1) there may be an epistemological relationship between how one learns and how one teaches, 2) pre-service elementary teachers may hold an idiosyncratic view that learning mathematics is an isolated endeavor, and 3) affective disposition toward mathematics may be related to a pre-service educator's ability to escape the trappings of the first two.

Theoretical Framework

Drawing methodology provides a means for appropriately eliciting memories, thoughts, and feelings of adults (Mitchell, Theron, Stuart, Smith, & Campbell, 2011). Since this study focuses on examining pre-service teachers' affective disposition, therefore, drawing was chosen as a methodological strategy. The idea that elementary teachers are not necessarily fond of mathematics is not new. Bulmahn and Young (1982) posed the same hypothesis over 30 years ago. They also asked the question of whether or not mathematics anxiety is akin to a communicable disease that can be passed from teacher to student. In either case, the suggestion that mathematics can be experienced by some students as "traumatic" was taken into consideration. As van Laren (2011) discovered, utilizing drawings allowed pre-service teachers to think about a sensitive topic in a "non-threatening, non-stressful" context (p. 143). Michell et al. (2011) suggest that allowing drawers to engage in verbal expression about their drawings allows for deeper and more complete meaning to emerge.

In these verbal expressions, it became practical to identify productive or nonproductive affective dispositional characteristics toward mathematics. For the purposes of this study, a productive disposition toward mathematics is defined by Kilpatrick, Swafford, and Findell (2001) as "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (p. 131). Honing in more specifically on the affective domain of disposition, we use a slightly modified list of characteristics from Beyers (2011): nature of mathematics, worthwhileness, usefulness, sensibleness, self-concept, attitude, and anxiety.

For the purposes of this pilot study, the characteristics of worthwhileness and sensibleness are relevant and therefore, further discussion is warranted. Worthwhileness as a characteristic within the affective domain of disposition toward mathematics refers to a person's supposed belief about whether or not spending time and effort in mathematics has an intrinsic and/or extrinsic reward. Sensibleness is a reflection of a person's belief that mathematics makes "sense" or can be understood.

Methodology

Among 5 sections of elementary mathematics methods courses, 139 undergraduate senior pre-service teachers completed two drawing tasks in two consecutive class sessions. The first session, they were presented with the task "Draw yourself learning mathematics," and in the second session, "Draw yourself teaching mathematics." Along with each drawing, participants were asked to "Describe and explain what you included in your drawing." Of the 139 participants, 30 were selected using a random number generator to comprise the pilot sample. However, 3 of these randomly selected participants were disqualified because they did not complete one of the two drawing tasks.

The random pilot sample consisted of 27 participants. Table 1 indicates demographic characteristics of the random pilot sample participants.

Table 1
Random Pilot Sample Participant Demographics

Gender	Female		Male	
Sample (n=27)	92.6%		7.4%	
Population (n=139)	90.6%		9.4%	
Race/Ethnicity	Hispanic/Latino	White/Caucasian	Other	
Sample (n=27)	74.0%	14.8%	11.1%[a]	
Population (n=139)	81.3%	6.5%	7.2%	
Languages Spoken	English-Spanish Bilingual	English Monolingual	Other	
Sample (n=27)	63.0%	33.3%	3.7%[b]	
Population (n=139)	78.4%	19.4%	2.2%	
Dominant Language	English		Spanish	
Sample (n=27)	70.4%		29.6%	
Population (n=139)	51.8%		46.0%	

Years Enrolled in US Schools	0-5	6-10	11-15	16-20	21+
Sample (n=27)	7.4%	3.7%	18.5%	51.9%	18.5%
Population (n=139)	10.8%	15.1%	18.0%	41.7%	12.9%

[a]One White-Hispanic-Italian, one Hispanic-White, and one White-Mexican.
[b]One English-Spanish-Italian Trilingual.

A total of 54 drawings underwent content analysis through open coding using NVivo 10 software. Initially, 102 codes were identified, which were then put into 11 categories. After this round of coding, each participant's pair of drawings was compared. Unique codes that appeared in one drawing but not the other were listed as well as codes that were identified in both drawings.

Findings

Among the 27 participants, 12 (44.4%) drew themselves teaching mathematics completely alone, as noted by the presence of the code "teacher alone." Meaning, in those 12 drawings, there were no students and the participant drew herself or himself as a teacher. The same 12 participants were among the 20 (74.0%) who drew themselves learning in isolation, coded "student alone." This means that in their drawings of themselves learning, they were the only people present. Of the 27 pairs of drawings, about one-third (n=8, 29.6%) did not share any identified codes between the two drawings. However, among these 8 participants, 6 drew themselves learning in isolation and teaching in isolation. Therefore, it could be argued that this link eliminates those 6 participants and only 2 (7.4%) participants have unique drawings between teaching and learning mathematics [EPS-F13-058, EPS-F13-127]. These two cases were investigated further, and the findings will be discussed later. Excluding these two cases, 25 (92.6%) of the participants created drawings with similar elements in both learning mathematics and teaching mathematics. An example of a participant's drawings with similar elements is illustrated in Figure 1. Once can see that the physical positioning of the learner is consistent in both drawings, as is the thought or speech bubble "I don't get it." Also of note is the question mark symbol (?) in the bubble in the learning drawing that is repeated in the teaching drawing near the participant's head.

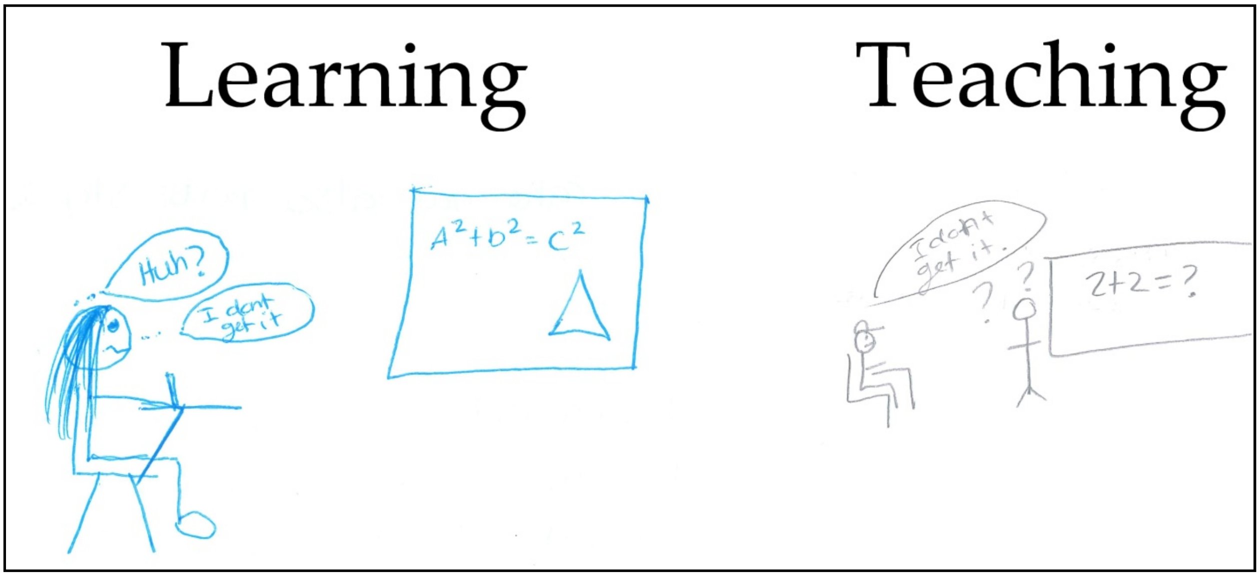

Figure 1. EPS-F13-051 learning and teaching mathematics drawings.

The two unique cases with divergent drawing characteristics mentioned in the previous paragraph [EPS-F13-058, EPS-F13-127] were investigated further. Figure 2 contains the learning and teaching drawings from participant EPS-F13-058. Looking at the two drawings, the differences in the content become apparent. The participant drew herself in isolation while learning, but included students in the drawing of her teaching. The "question mark" and "light bulb" symbols in the learning drawing are absent from the teaching drawing. However, it could be argued that the students represented in the teaching drawing may be engaged in isolated learning. An interview with the participant may be useful to obtain this kind of verification.

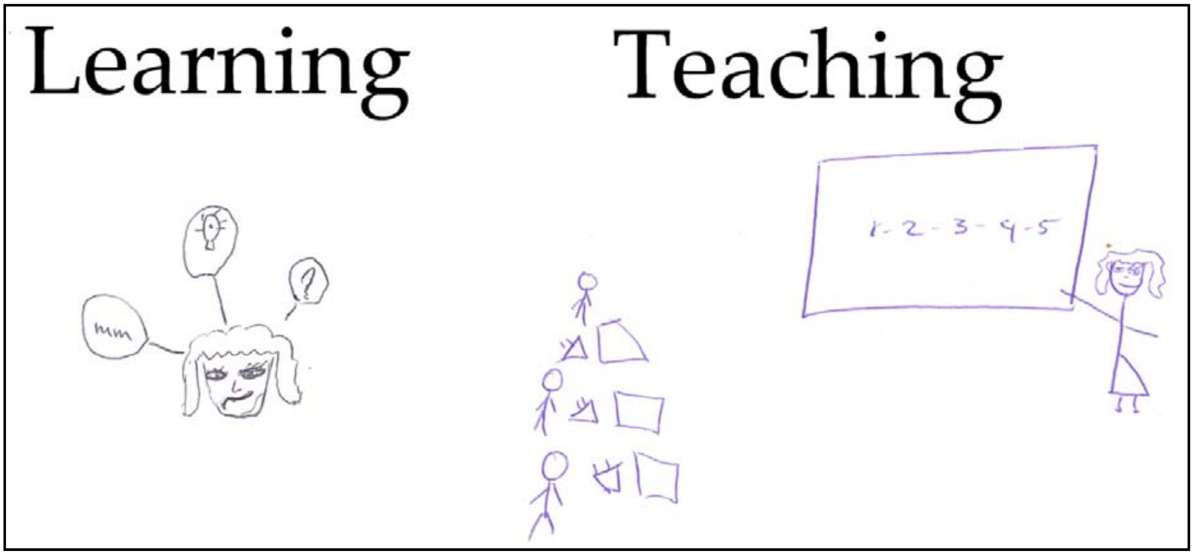

Figure 2. EPS-F13-058 learning and teaching mathematics drawings.

Figure 3 includes are the learning and teaching mathematics drawings from participant EPS-F13-127. As has been noted in Figure 2, drawings in Figure 3 also portray the participant learning in isolation while including students in the teaching mathematics drawing. Similarly, the use of “question mark” and “exclamation point” symbols in the learning drawing are absent in the teaching drawing. However, in the teaching mathematics drawing, it could be argued that the students are not learning in isolation because they are drawn at tables facing each other, which could imply they are learning collaboratively. Again, an interview with the participant may be an effective strategy to verify this conjecture.

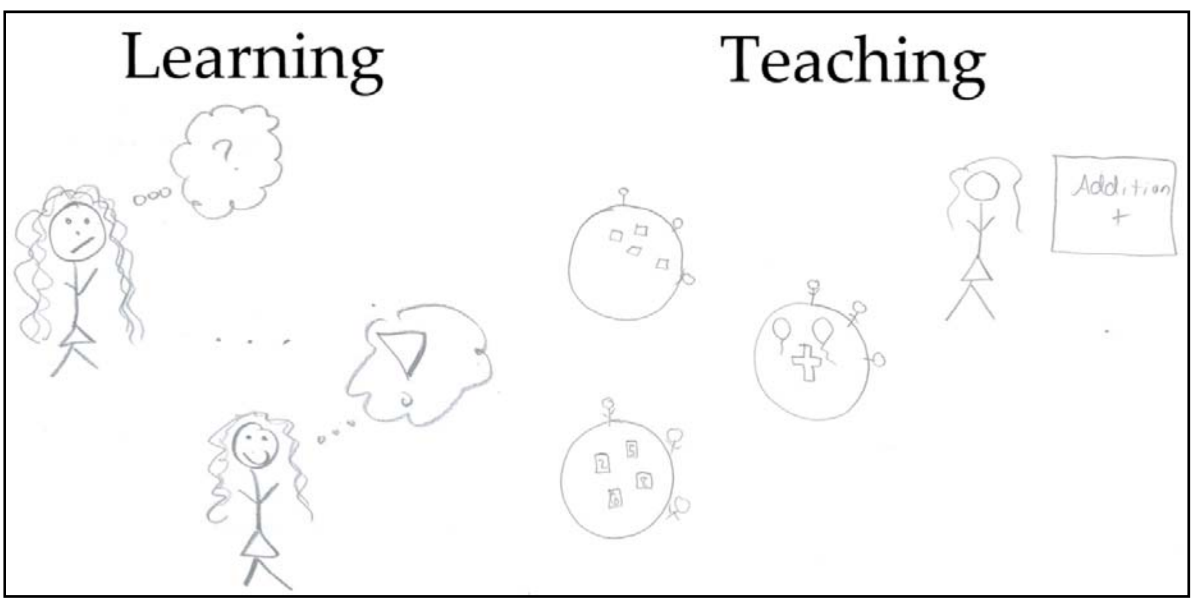

Figure 3. EPS-F13-127 learning and teaching mathematics drawings.

The participants were also asked to write both a description and explanation of their drawings. Mitchell et al. (2011) suggest drawings be complemented by some form of verbal data. This way, the participant has the opportunity to articulate the intended meaning(s) of their drawings. Of particular intrigue is the comparison of the written elements for the learning drawing between these two participants. Figure 4 represents the two participants' writings. In each case, the participants use language that suggests a productive affective disposition toward mathematics, specifically with respect to the characteristics of worthwhileness and sensibleness. Both participants indicated that they endure a "struggle" while learning mathematics, but identify an intrinsic payoff after enduring their struggle (using the words "happy" and "enjoy"). At the same time, both participants express that mathematics is something that can be understood. For example, EPS-F13-058 states "I feel happy when I understand" and EPS-F13-127 "However when it 'clicks' and makes sense I enjoy figuring out problems."

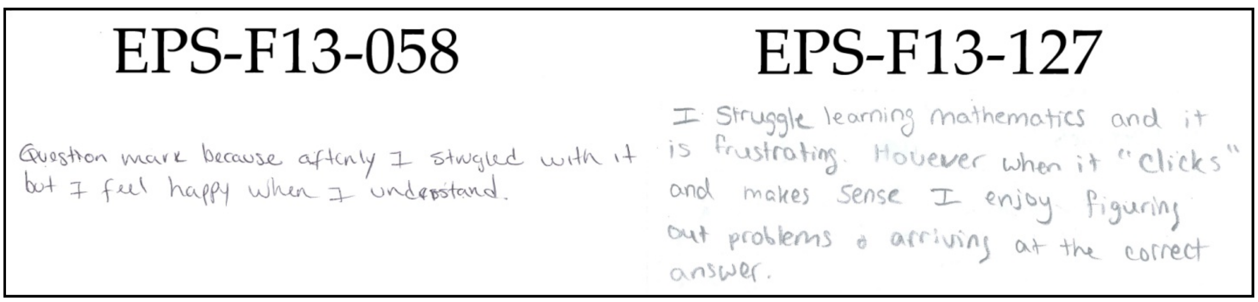

Figure 4. EPS-F13-058 and EPS-F13-127 learning mathematics descriptions.

Discussion

The majority (92.6%) of participants provided drawings that incorporated similar elements in for both teaching and learning mathematics. It is possible that this could be attributed to the theory that we teach how we learn, which was posited by Handal (2003), but requires further investigation. A smaller majority (74.0%) drew themselves learning in isolation, which may be explained by the participants' epistemological stance that mathematics learning and knowledge is arrived at through an individual process, in isolation and not socially, through collaboration.

By contrast, the two participants in this random pilot sample did not incorporate the same elements in their two drawings, but did draw themselves learning in isolation. Perhaps being able to separate the way one teaches from the way one learns requires having a productive affective disposition toward mathematics, in this case perceiving mathematics as both worthwhile and sensible. This would support Cross' (2009) assertion that beliefs about the nature of mathematics (only one of seven distinct characteristics that comprise affective disposition) are consistent with views on student learning and mathematics teaching. However, further data collection and analysis will be needed to verify these conjectures. It is quite possible that EPS-F13-058 carries the epistemological stance that learning mathematics is an isolated endeavor into her teaching philosophy. Subsequent drawings and participant interviews may shed light into this issue.

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DEVELOPING PRESERVICE MATH TEACEHRS'DIVERSITY AWARENESS AND KNOWLEDGE

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This report presents results from recent work of a design experiment that employs strategies to enhance middle grades preservice teachers' knowledge and beliefs about teaching algebra for equity. The participants were 57 middle grade mathematics preservice teachers enrolled in a required Mathematics Problem Solving course. We address the question: What instructional activities contribute to growth in problem solving, teaching problem solving, and cultural beliefs? The findings indicate that course activities improved participants' algebra knowledge and their knowledge about teaching for equity. Growth was non-linear, reflecting initial confidence, then realizing their limitations, and finally regaining confidence in their new knowledge.

Literature Review

One means to improve the mathematics knowledge and skills of students is to enhance the capacity of middle grades teachers (Hill, Rowan, & Ball, 2005). The nature of and need for improving teachers' mathematics knowledge is well-established (Huang & Kulm, 2012, Kulm, 2008; National Mathematics Advisory Panel, 2008). Ponte and Chapman (2008) suggested that mathematics curricula and instructional approaches should be enhanced to address deficiencies in prospective teachers' mathematics knowledge. Hiebert and Morris (2012) have recently stated that to improve classroom instruction we should shift our attention from the recruitment of talented and qualified people to the improvement of the instructional methods that are implemented in classrooms (p. 92).

In addition to the necessary mathematics knowledge for teaching (Kulm, 2008), today's culturally and ethnically diverse middle grades classrooms require teachers with the equity consciousness (McKenzie & Skrla, 2011) necessary to be effective. To prepare for a multicultural school environment, it is equally important to address the mathematics preparation and the beliefs and perceptions of the preservice teachers (Ladson-Billings, 2011). This preparation is especially important since often the lowest performing students belong to a racial, ethnic, income, or cultural group different from the prospective teachers themselves, who are overwhelmingly female, white, and middle class (Sleeter, 2001). Many preservice teachers are not prepared to teach in a multicultural setting. Watson, Charner-Laird, Kirkpatrick, Szczesiul, and Gordon (2006) stated that "very few teacher education programs have successfully tackled the challenging task of preparing teachers to meet the needs of diverse populations'' (p. 396). Indeed, a recent study by Nelson and Guerra (2013) found that even though the majority of teachers and educational leaders in two school districts in Texas and Michigan seem to be aware of culture, they neglected obvious aspects of culture and still used deficit thinking.

Few research projects have tried to address in a holistic manner the mathematics knowledge, pedagogical content knowledge as well as equity and multicultural awareness of the preservice teachers (Anderson, 2013). This paper presents the results from the work of a 5-year NSF-funded design experiment that employs several strategies to enhance middle grades preservice teachers' knowledge of teaching algebra problem solving for equity. Brown, Davis, Lewis, and Kulm (2011) confirmed that integrating instruction in problem solving and activities to promote preservice teachers to think about teaching for diversity was effective in enhancing beliefs about teaching for equity. The preservice teachers exhibited a non-linear, positive pattern in the development of their awareness of teaching for diversity (Merchant, Kulm, Davis, Ma, & Oner, 2013). Preservice teachers' judgment of cultural relevance of algebra problems was focused on whether or not students might have had direct, personal experience with the specific context of a problem in their own lives. However, most of them learned to use concrete representations, engage the students in discussion, and ask questions that revealed the student's source of misunderstanding (Kulm, Merchant, Ma, Oner, Davis, & Lewis, under review). In the current paper, we investigated the effectiveness of specific types of activities on these improvements in order to answer the research question: What are the effects of math problem solving and diversity activities on preservice teachers' awareness and knowledge about (a) teaching for diversity, (b) algebra problem solving, and (c) teaching algebra problem solving?

Methods

This research is focused on a required Mathematics Problem Solving course for middle grades math teacher certification. The course has been revised over a period of five semesters to include activities and assignments to address issues of diversity and culture in teaching algebra. The design of the course includes four primary, interrelated components: (1) Math Problem Solving and Problem Posing, (2) Math Problem Equity Challenges, (3) Readings and Discussions on Diversity, and (4) *Second Life* (*SL*) Tutoring and Teaching. This report focuses on the impact of the first three sets of course activities. We have reported elsewhere on some of the effects of *Second Life* activities (Davis, Chien, Brown, & Kulm, 2012).

Participants. The 57 participants included 52 females and 5 males. There were 45 White females, 2 African-American females, 3 Hispanic females, 5 White males, with 2 not responding.

Procedures. The authors were the instructor and course assistants for the two semesters that comprise this study. The following summary provides a brief summary of the course activities.

- Math Problem Solving and Problem Posing. Instruction and practice with problem solving heuristics, using *How to Solve It* (Polya, 2004) as the primary textbook. Students completed several problem sets that applied heuristic methods, writing out complete solutions.
- Math Equity Problem Challenges. Each of three assigned Equity Problem Challenges consisted of four components: a culturally relevant problem to solve and adapt for middle grade students, responding to student misconceptions, planning a problem solving lesson, and answering mathematics and equity questions that middle grades students might ask.
- Readings and Discussions on Diversity. Assigned readings were given from the textbook, *Responding to Diversity* (Ellis, 2008) and other essays on teaching for diversity. Two guest lecturers presented and discussed cultural diversity and cognitive engagement.

Data Sources. Participants completed pre- and posttests of the Knowledge for Algebra Teaching for Equity (KATE) test developed by the authors. The KATE test contained 20 Likert items (Strongly agree, Agree, Disagree, Strongly disagree) adapted from the Cultural Awareness and Beliefs Inventory (CABI) (Roberts-Walter, 2007), and 19 open-ended mathematics problems to assess algebra content and teaching knowledge. The Cronbach alpha reliability of the CABI is .77 (Anderson, 2013). Participants also completed a Diversity Preparedness Response Inventory (DPRI), developed by the authors to capture students' responses to the impact of the specific activities in the course. Participants completed the DPRI twice during the semester, using a four-point scale (No Change, Made me rethink, Changed somewhat, Changed a lot).

Data Analyses. To address the construct validity of the CABI and DPRI, exploratory factor analysis (EFA) was used. Varimax rotation was employed, which is used in principal component analysis and is an orthogonal rotation that shows uncorrelated factors. The correlation between predictors and outcomes was calculated, and linear regression analysis and polynomial regression were used to determine the effect of predictor variables on outcome variables.

Findings

Exploratory Factor Analysis of the CABI revealed four factors: Teacher Efficacy, Teaching Beliefs, Culture, and Racial Beliefs. Table 1 provides example items for each of these factors.

Table 1. *Factors and Sample Items from the CABI Instrument.*

Factor	Number of items	Sample items
Teacher efficacy	11	I am comfortable with people who exhibit values or beliefs different from my own.
Teaching beliefs	4	I believe that poor teaching is the main factor that causes the gap in math achievement between White students and students of color.
Cultural beliefs	3	I believe students in poverty are more difficult to teach.
Racial beliefs	2	I believe many middle school teachers engage in biased behavior toward students of color in the classroom.

The EFA analysis of the DPRI resulted in two factors: Teaching Problem Solving Knowledge (TPSK) and Teaching for Equity Knowledge (TEK). Table 2 shows the factor loadings.

Table 2. *Rotated Component Matrix of Diversity Preparedness Response Inventory.*

Class Activities and Assignments	TEK	TPSK
Readings from Polya's *How to Solve It* textbook	.014	.866
Readings and questions from *Responding to Diversity* textbook	-.012	.899
Presentations and discussions about culturally relevant problems	.065	.904
Diversity readings Ladson-Billings and Milner articles	.828	.028
Guest speaker presentation – Culturally Relevant Teaching	.816	.044
Human Graph Math Equity Challenge Problem	.827	.184
Analysis of Language Moves from tutoring	.778	.085
Guest speaker presentation – Cognitive Engagement	.598	-.086
Food Drive Math Equity Challenge Problem	.742	.162
Dinner Problem Math Equity Challenge	.235	.893

It is worth noting that two of the equity challenge problems; Human Graph and Food Drive were part of the teaching for equity factor, while the Dinner problem was part of the teaching problem solving factor. The readings from the *Teaching for Diversity* (Ellis, 2008) text also were a part of the teaching problem solving factor, indicating that the participants focused less on the diversity ideas than the teaching strategies that were presented in the text. The KATE instrument was composed of two factors: Math, which was made up of questions that addressed understanding and problem solving skills in algebra and Teach, which was comprised of questions that asked participants how they would address students' errors and misconception in understanding and solving algebra problems. Because there were three different surveys, we calculated average scores to obtain the factor scores. In order to address the research questions, we first calculated the intercorrelation coefficients between all of the variables: the DPRI factors which were the independent variables and the CABI and KATE factors which were the dependent variables (see Table 3).

Table 3. *Intercorrelations of All Variables.*

	Teacher Efficacy	Teaching Beliefs	Culture	Racial Beliefs	Math	Teach	TPSK	TEK
Teacher Efficacy	1.000							
Teaching Beliefs	.538	1.000						
Cultural Beliefs	.354	.376	1.000					
Racial Beliefs	-.860	-.227	-.047	1.000				
Math	-.001	.059	-.097	.104	1.000			
Teach	.084	-.030	-.104	.234	.576	1.000		
TPSK	.023	-.004	.120	.059	-.125	.196	1.000	
TEK	.341	.440	.171	-.113	.141	.126	.168	1.000

To better understand the relationship between independent and dependent variables, we first tried linear regression analyses, which showed that there was not a statistically significant linear relationship between the variables. We then explored the distribution of the six dependent positively or negatively, raising the possibility of a polynomial relationship between independent and dependent variables. A nonlinear regression analysis produced a quadratic relationship between the variables. Table 4 shows the significant quadratic relationships between independent and dependent variables.

Table 4. *P Values and Effect Sizes for Quadratic Models.*

Independent Variable	Dependent Variable	*p* value	Effect size (R^2)
Teaching for Equity Knowledge	Math	.007	.169
Teaching for Equity Knowledge	Teaching Efficacy	.006	.170
Teaching for Equity Knowledge	Teaching Beliefs	.001	.224
Teaching Problem Solving Knowledge	Teaching Beliefs	.027	.125

Although the effect sizes are small, the relationships provide some relevant information related to the research question. The most effective activities were those that aimed at knowledge about teaching for equity. The activities that participants believed changed their ideas about teaching for equity appeared to have the most widespread impact on their knowledge about math, their feelings about efficacy, and their attitudes and beliefs about teaching for diversity. The activities related to teaching problem solving had an impact only on their teaching beliefs.

Discussion

The activities employed in the course appear to provide an effective combination of approaches that improve preservice middle grade math teachers' knowledge and beliefs about teaching for equity. Some activities are more effective than others and they have impacts on different aspects of knowledge and beliefs. It is important to note that our central activity, the Equity Challenge problems, improved both content knowledge and knowledge about teaching diverse student. Since the Dinner Problem challenge was the first one presented, the participants may have seen it most related to teaching problem solving. Participants perceived the later challenge problems as changing their ideas about diversity.

The quadratic pattern confirms our informal observations and previous research (Merchant, et al., 2013) that participants often enter the course with positive beliefs and confidence about teaching diverse students. As they encounter the activities and assignments, they recognize their limitations, then at the end become more confident again, armed with the knowledge and practice gained. These results also confirm the long-term efforts required to affect attitudes and beliefs about teaching mathematics for diversity, as well as the uneven progress that should be expected of most preservice teachers.

These results have some limitations that affect making generalizations. The small sample size and exploratory nature of the analyses should be noted. The idea of combining instruction in mathematics problem solving with attention to diversity was new to the participants and it was not always clear that all of them "bought into" the approach in what they expected to be mainly a mathematics course. On the other hand, most participants appreciated the opportunity to learn about teaching strategies and to practice lesson planning within a mathematics course. The research team believes that this integrated approach is necessary to prepare prospective teachers for today's diverse middle grades classrooms and will continue to refine the course activities in light of these results.

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SECONDARY MATHEMATICS PRESERVICE TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL THINKING

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The purpose of this study is to examine how secondary mathematics preservice teachers attend, interpret, and respond to twelfth grade students' mathematical thinking of extending a pattern. Utilizing the Kruskal-Wallis test, significant differences were found between how sophomore and senior level participants, as well as junior and senior level participants respond to students' written work. The results of this study will help inform the current courses of study and practices within a teacher education program located at a large university.

Theoretical Framework and Related Literature

The National Council of Teachers of Mathematics (NCTM) has identified a component of effective teaching as "observing students, listening carefully to their ideas and explanations...and using the information to make instructional decisions" (NCTM, 2000, p. 19). The teacher's role in facilitating learning relies upon his/her ability to monitor student thinking in the moment. It is also reliant on his/her ability to help students construct understanding that is connected to their prior knowledge and that is appropriately situated within the topic under consideration. When communication in the classroom challenges and supports student thinking, students' ability to logically explain and deduce mathematical concepts is developed (Peressini, Borko, Romagnano, Knuth & Willis, 2004). Furthermore, communication and instruction can support learning when teachers are able to effectively attend to students' thinking, noting specific strategy use, as well as when teachers appropriately interpret and respond to students' thinking in ways that help students extend their understanding (Hiebert, Morris, Berk, & Jansen, 2007; Jacobs, Lamb, & Philipp, 2010).

Studies suggest that the ability to consider students' mathematical thinking can be learned and that teachers can become more responsive to students' thinking when they are provided with opportunities to examine students' written work and to collaboratively consider the ways in which they might respond to this work so that students' understanding is extended (Bright & Joyner, 2005; Doerr, 2006). A teacher's ability to respond to students' written work is largely dependent on their content knowledge, pedagogical knowledge, pedagogical content knowledge (Shulman, 1986), and knowledge about students' mathematical thinking and learning (Even &

Tirosh, 1995; Hill, Ball, & Schilling, 2008). Also relevant to this study is the concept of professional noticing, defined as "how, and the extent to which, teachers notice children's mathematical thinking" (p. 171), more specifically, the framework of *professional noticing of children's mathematical thinking* (Jacobs et al., 2010). This three-part framework is composed of the following interrelated skills: (a) *attending* to student's solution strategies, (b) *interpreting* students mathematical understanding, and (c) deciding how to *respond* to students' understanding. The construct was utilized by the researchers to examine the manner in which secondary mathematics preservice teachers unpack twelfth grade student's mathematical thinking of extending a pattern of tiles. It was helpful for the researchers to consider the elements (attending, interpreting, and responding) distinctly, but it should also be noted that teachers do not, in the moment, consciously differentiate between these factors. Instead, teachers may move freely between these components, constantly making sense of students' thinking and making instructional decisions based on the students' understanding. Furthermore, the researchers considered four categories of responses that preservice teachers used when responding to the students' written work: general move, child's affect, teacher's mathematical thinking, or child's mathematical thinking (Jacobs & Philipp, 2010).

Recent studies conducted on in-service elementary teachers have suggested that classroom instruction that builds upon and connects to students' mathematical thinking has been associated with classroom interactions and discourse that promote collaboration and conceptual understanding (Cobb et al., 1991), an increase in students' mathematical achievement (Fennema et al., 1996), and a change in teacher's pedagogical beliefs (Cobb et al., 1996) and beliefs about teaching and learning in general (Fennema et al., 1996). Similar research at the secondary and at the preservice level is more limited (Heid et al., 1998; Henningsen & Stein, 1997). Furthermore, the majority of research on professional noticing has been conducted through professional development efforts (Choppin, 2011; Jacobs et al., 2010) and through reflecting on classroom videos (Seidel et al., 2011; Sherin & van Es, 2009). There is also a vast amount of research that examines what is being noticed (Miller & Zhou, 2007; Star & Strickland, 2008) and does not examine why preservice and in-service teachers notice certain aspects of the classroom and student work or the ways in which they use what they notice to inform their instructional decisions. The present study begins to fill in the gaps in the research by examining the ways in which secondary mathematics preservice teachers attend, interpret, and respond to students' mathematical thinking of problems that involve extending a pattern. More specifically, the following research question was explored: How do secondary mathematics pre-service teachers attend, interpret, and respond to students' mathematical thinking? Currently, examining students' written work as a way to gauge students' mathematical understanding is not a specific topic of study at this university. We anticipate that the results of this study will inform courses of study at the undergraduate and graduate levels.

Methodology

Participants were recruited using a convenience sampling approach. Researchers contacted all sophomore, junior, and senior undergraduate and Secondary Master of Arts in Teaching (MAT) graduate preservice mathematics teachers at a large Southeastern university located in the United States and asked for volunteers to participate in the study. Participants (n = 30; 25 females, 5 males) included 6 sophomores, 12 juniors, 9 seniors, and 3 MAT students. Researchers offered participants distinct times to assist with the study; participants only attended one 30-45 minute session. During this time, each participant was provided with three samples of student work, each highlighting a different approach to one mathematics problem. The mathematics problem, with accompanying student work, was selected from the National Assessment of Educational Progress Exam (Brown & Clark, 2006).

Participants were asked to analyze the three student work samples and respond to the prompts found below for each sample. This process produced nine responses per participant, three for each student work sample.

1. Please describe in detail what you think this student did in response to this problem.
2. Please explain what you learned about the student's understanding.
3. Pretend that you are the teacher of this student. What problem or problems might you pose next and why?

Prior to coding, the researchers discussed and defined how each participant's response would be scored based on the scale employed by Jacobs and colleagues (2010). Using initial coding, a rubric was developed and the researchers coded one participant's responses, while simultaneously discussing and refining the codes. The researchers coded two more participant's responses separately and established inter-rater reliability at 89%, finalizing the coding scheme. The researchers then utilized the rubric to code independently the remaining responses from each participant.

Participant responses to Question 1 were scored either a 0 for lack of evidence or a 1 for proof of evidence of *attending* to the student's thinking. Responses to Question 2 and Question 3 were scored a 0 for no evidence, a 1 for partial evidence, or a 2 for rigorous evidence for proof of *interpreting* the student's mathematical approach and for proof of *responding* to the student's strategy and understanding by posing a subsequent problem(s). The participants received a median score for each component of professional noticing. Additionally, participants' responses to the third question, Pretend that you are the teacher of this student. What problem or problems might you pose next and why?, were coded as either child's mathematical thinking, teacher's mathematical thinking, child's affect, or general teaching moves (Jacobs & Philipp, 2010). Categorizing participant responses in this way helped researchers gauge the support or lack thereof provided by participants when responding to students during or after solving a problem. Kruskal-Wallis tests were conducted to examine if statistical differences ($\alpha = .05$) exist in responses based on participant's year in the teacher education program. If a statistical difference was found, the Mann-Whitney U test was employed to evaluate pairwise differences among the four groups. To control for Type I error, the researchers used the Bonferroni correction ($\alpha = .008$).

Findings and Discussion

The percentages of participant responses in each area of noticing are displayed in Table 1. Findings suggest that the MAT and senior level participants tend to be more adept at *attending* to students' mathematical thinking. This might be attributed to the fact that these participants have obtained an undergraduate degree in mathematics or have completed their required mathematics coursework, respectively. In *interpreting* students' work, the majority of participants exhibited partial evidence. Participants had a tendency to focus only what the student did understand; most participants did not consider what the student did not understand. Findings from the *respond* category suggest that three-fifths of the participants created a problem that did not connect to the students' work; the problem could have been created without first examining the students' response. However, the majority of senior level participants responded to students strategies by posing a problem that extended students' understanding and/or encouraged the use of another approach to solving the problem. In conducting Kruskal-Wallis tests, significant difference was found for how participants responded to the students' written work ($\chi^2(3, N=30) = 15.0375$, $p = .0018$), but not for how participants attended to ($\chi^2(3, N=30) = 3.7927$, $p =$

.2847) or interpreted (χ^2(3, N =30) = 1.2587, p = .7390) the students' written work. The follow-up analyses revealed that sophomore level participants (χ^2(2, N =15) = 14.449, p = .0007) and junior level participants (χ^2(2, N =21) = 12.983, p = .0015) differed significantly from senior level participants. It could be hypothesized that a higher percentage of senior level preservice teachers exhibited rigorous evidence of posing a follow-up problem due to their accumulated experiences in the field and current enrollment in a mathematics methods course.

Table 1 *Number of Participants (Percentage) at each Level of Evidence*

Component	**2nd year (n = 6)**	**3rd year (n = 12)**	**4th year (n = 9)**	**MAT (n = 3)**	**Total (N = 30)**
		Attend			
Proof of evidence	3 (50%)	6 (50%)	7 (78%)	3 (100%)	19 (63%)
Lack of evidence	3 (50%)	6 (50%)	2 (22%)	0 (0%)	11 (37%)
		Interpret			
Rigorous evidence	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Partial evidence	4 (67%)	6 (50%)	7 (78%)	1 (33%)	18 (60%)
No evidence	2 (23%)	6 (50%)	2 (22%)	2 (67%)	12 (40%)
		Respond			
Rigorous evidence	0 (0%)	0 (0%)	2 (22%)	1 (33%)	3 (10%)
Partial evidence	0(0%)	2 (17%)	7 (78%)	0 (0%)	9 (30%)
No evidence	6 (100%)	10 (83%)	0 (0%)	2 (67%)	18 (60%)

Table 2 displays the four categories of responses that participants provided for supporting students' problem solving. As can be seen, the majority of participants provided a general move in response to students' mathematical thinking, which indicated a lack of specificity and a response general enough to be relevant for any student. This result is not surprising when considering the percentage of participants who scored zero or exhibiting no evidence on the respond category (see Table 1). No participants responded in a way that would support the affective characteristics of the student (e.g., confidence or self-esteem). We hypothesize that this may be attributed to the limited experiences that the preservice teachers have working closely with students prior to student teaching. Additionally, approximately one-fifth of the participants provided a response focused on the teacher's mathematical thinking, which suggests that the participants focused on how they might solve the problem to obtain a correct answer. Another one-fifth of the responses were focused on the child's mathematical thinking, building upon and challenging the student based on his/her understanding of the problem. Specifically, undergraduate students in their fourth year account for half of these total responses. Furthermore, a significant difference was found to exist between participants that responded using general moves ($\chi^2(3, N=90) = 9.5414$, $p = .0229$). The follow-up tests concluded a statistical difference exists between junior level and senior level participants ($\chi^2(3, N=21) = 13.634$, $p = .0034$). We hypothesize that this decrease in responding in a general way to students, from the third year of undergraduate study to the fourth year, may be credited to the increased focus on teacher education mathematics coursework and increased field experiences during the senior year.

Table 2 *Number (Percentage) for reasons for supporting students' mathematical thinking*

Reasons	**2nd year**	**3rd year**	**4th year**	**MAT**	**Total (N = 90)**
General Move	12 (66%)	27 (75%)	8 (30%)	7 (78%)	54 (60%)
Child's Affect	0 (0%)	0 (0%)	0 (0%)	0 (0%)	0 (0%)
Teacher's Mathematical Thinking	3 (17%)	7 (19%)	10 (37%)	0 (0%)	20 (22%)
Child's Mathematical Thinking	3 (17%)	2 (6%)	9 (33%)	2 (22%)	16 (18%)

Future Research

To further explore the ways in which secondary mathematics preservice teachers notice a student's mathematical thinking through an analysis of the student's written work, we are developing and extending this study to include an interview component. Interview transcriptions will be coded to determine what patterns in responses exist. We will also engage participants in another analysis opportunity. We hypothesize that this repeated opportunity will lead participants to demonstrate increased evidence in the ways in which they attend, interpret, and respond to students' mathematical thinking. Furthermore, we plan to examine course syllabi to determine existing structures and gaps in current programs related to preservice teachers' understanding of students' mathematical thinking. We anticipate that this examination of syllabi and the results of this study will help inform the current courses of study and practices within a teacher education program located at a large university.

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PRE-SERVICE TEACHERS' CONCEPTIONS OF REPRESENTATIONS OF EQUIVALENT FRACTIONS AND OF FRACTION UNITS

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This article reports on a qualitative investigation of forty-two pre-service elementary school teachers' ability to find and demonstrate equivalency of fractions using diagrams and on conceptions of representations of discrete and continuous wholes. Although a majority of the participants were able to find equivalent fractions, classify quantities as being discrete or continuous, and represent a fraction with a continuous whole, only a few successfully demonstrated equivalencies of fractions using diagrams and a only a few correctly represented a fraction of a discrete whole. Representations and explanations provided by some of the participants and implications on teacher education are discussed.

Diagrams and symbolic expressions are examples of mathematical representations (NCTM, 2000), and are vehicles for learning and communicating (Friedlander & Tabach, 2001). Mathematical argumentation requires one to talk or write revealing their reasoning to peers and teachers. This should be taught and learned in classrooms (Lampert & Cobb, 2003).

Mathematical knowledge for teaching is either subject matter knowledge (SMK) or pedagogical content knowledge (PCK) (Shulman, 1986). PCK has four components, one of them being the knowledge of instructional strategies which encompasses explanations, illustrations, and representations (Grossman, 1990).

A number of studies (e.g. Southwell & Penglase, 2005; Thompson & Thompson, 1996; Ball, 1990) have revealed deficiencies in fraction knowledge among prospective and in-service teachers and studies specifically investigating prospective teachers' conceptions about operating with fractions abound (e.g. Ball, 1990; Huang, Liu, & Lin, 2009; Rizvi & Lawson, 2007; Son & Crespo, 2009). This study adds to the literature by investigating pre-service teachers' conceptions and use of fraction representations. Specifically, the study investigated pre-service teachers' (a) conceptions about discrete and continuous wholes and (b) ability to find equivalent fractions and demonstrate equivalence with diagrams. Learning about the understandings prospective teachers bring to teacher education can help inform efforts to build prospective teachers' content knowledge to levels needed for teaching, levels which may not be attained through teaching itself (Ball, 1988).

Literature Review

Fraction concepts are among the most challenging topics learnt in elementary school (Siebert & Gaskin, 2006). Difficulties with fractions are due to different meanings of fractions (Sonnabend, 2010) and students' failure to connect representational forms of fractions and real-world situations (Hiebert, 1985). A solid understanding of different representations of fractions, decimals, and percents is necessary for working flexibly with rational numbers (NCTM, 2000).

Representations used by teachers influence students' representation choices which in turn impacts problem solving (Cai & Lester, 2005). Inaccurate use of words by teachers can lead to student errors, for example students writing $2/3 \times 2 = 4/6$ because the teacher referred to process of obtaining $4/6$ from $2/3$ as doubling (Ding, 2008). Research (Jigyel & Afamasaga-Fuata'I, 2007) has shown that indeed some students think that $4/6$ is double $2/3$. Teachers need to pay attention to verbal and textual representations they use in the classroom because student misconceptions may result from the use of some representations (Muzheve & Capraro, 2012). It is worth pointing-out that fluency in mathematical terminology allows students to read, understand, and discuss mathematical ideas. Words and phrases habitually taken for granted by teachers are often foreign and challenging to students (Thompson & Rubenstein, 2000).

Despite the fact that fractions are introduced in the 4th grade and encountered repeatedly in subsequent grades, a majority of students, even those in the 6th grade, do not thoroughly understand equivalent fractions due to improper teaching of equivalent fractions and common denominators (Kamii & Clark, 1995).

The cited research on the importance of understanding different representations of fractions and the research on how teacher representational choices impact students' representational choices and the connections into teachers' explanations and students' conceptions provide a framework for collecting and analysis the data in this study.

Methods

Forty-two participants were recruited on a self-selected basis from two groups of elementary education students registered for the first part of a two-series mathematics course. Data were collected using a four question instrument and through follow-up questions and discussions with a purposely selected subset of the participants. Follow-up questions and discussions served as a way for obtaining additional information and data for triangulation purposes.

Instrument items

1) Which label, discrete or continuous, applies to each of these "wholes"? (a) mileage traveled by a wagon train (b) number of cartons of milk ordered for snack time in kindergarten.
2) Make a drawing that shows (a) $2/5$ of a continuous whole (b) $2/5$ of a discrete whole.
3) (a) Find a fraction equivalent to $3/5$ and (b) demonstrate with a diagram that the two fractions are equivalent.
4) Write the simplest form for $450/720$.

Each participant's responses were coded 1 if correct and 0 if wrong. To obtain an estimate of coding reliability, a knowledgeable other recoded 25% of the responses. There was 100% agreement on coding the responses. Notes on representations used for items 2(a), 2(b) and 3(b) and on procedures used by participants in obtaining the simplest form for $450/720$ were stored in to a spreadsheet for analysis.

Results

Table 1

Number of Correct Responses to each question

Question	1(a)	1(b)	2(a)	2(b)	3(a)	3(b)	4
Students who had not seen an example of discrete/continuous whole	16	15	14	1	17	8	16
Students who had seen an example of discrete/continuous whole	20	20	19	14	23	16	24
Total	36	35	33	15	40	24	40

Thirty-six and 35 of the 42 participants correctly classified mileage and the number of cartons as being continuous and discrete respectively. Some of the few students who incorrectly answered these two questions either guessed or reversed the meanings of the terms. In deciding whether the number of cartons represented a discrete or continuous whole, at least two students erroneously considered the amount of milk instead of considering the number of cartoons.

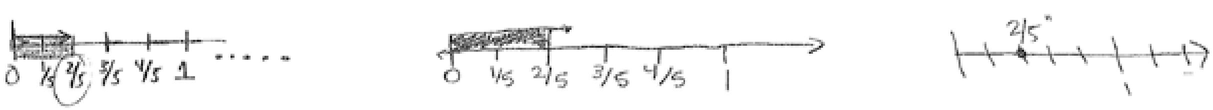
Thirty-three participants correctly represented $2/5$ of a continuous whole and only about $1/3$ correctly represented $2/5$ of a discrete whole. Only one of the 17 students who had not seen examples of representing fractions of discrete or continuous wholes correctly represented $2/5$ of a discrete whole. More than half of the participants, who had seen such examples, correctly answered the question. A majority of those who incorrectly represented $2/5$ of a discrete whole thought the whole they used was discrete because they could count the number of parts in the subdivided continuous whole.

Thirteen of the 15 students who correctly represented $2/5$ of a discrete whole did so with combinations of circles, squares, and triangles, but without any boundary around them (see Figure 1(a)). The other three used a representation like the one shown in Figure 1(b), with the boundary around the circles indicating the five circles constitute the whole.

(a) (b)

Figure 1. *Representing* $2/5$ *of a discrete whole*

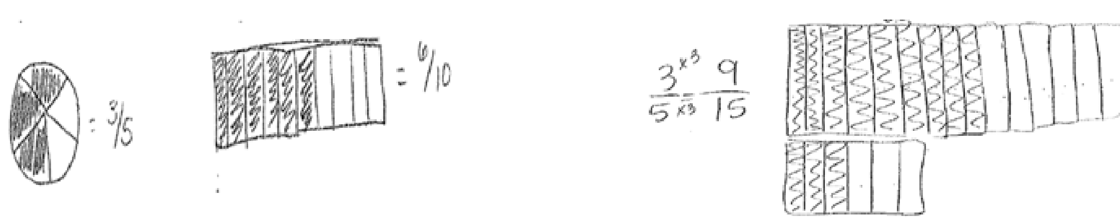
Some participants who had not seen examples of representing fractions of discrete and continuous wholes represented $2/5$ of a continuous whole as in Figure 2. The reasoning behind the use of dots in Figure 2(a) or arrows in Figures 2(b) and 2(c) was that the whole is continuous and therefore extends forever.



(a) (b) (c)

Figure 2. *Representing* $2/5$ *of a continuous whole*

Although 93% participants found a fraction equivalent to $3/5$, only 56% correctly demonstrated equivalence using a diagram. Common errors were using different shapes for the wholes or using wholes of different sizes as in Figures 3(a) and 3(b).



(a) (b)

Figure 3. *Attempts to demonstrate equivalence of two fractions*

Ninety-three percent (93%) of the participants correctly expressed $\frac{450}{720}$ in simplest form even though about 10% divided with a whole number as shown in Figure 4.

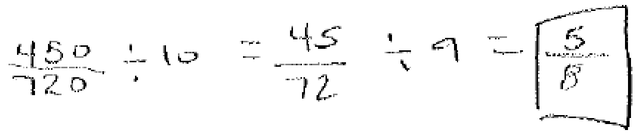


Figure 4. *Process of finding simplest form*

Discussion

The terms discrete and continuous are not commonly understood despite the abundance of and encounters with discrete and continuous quantities in everyday life. In Question 1(b), erroneously focusing on the amount of milk instead of focusing on the number of cartoons led some participants to the conclusion that the whole is continuous. We note that not all fractions make sense for every discrete whole. For example, half of a whole that consists of say seven pizzas is three and a half pizzas. It would not make sense to talk about half of a whole of say seven people.

Incorrectly representing $2/5$ of a discrete whole was mostly due to a lack of an understanding of that continuity of a whole is dictated by the number of entities that make-up the whole. Although correctly associating discrete with counting, participants missed another crucial aspect of discreteness; a whole made-up of at least two non-touching entities. In representing $2/5$ of a continuous whole, some participants used a number line or representations suggesting the whole should extended indefinitely. The understanding arises from the word continuous being synonymous with both nonstop and unbroken, the latter being the meaning that should be used. When used in the context of fractional wholes, the terms discrete and continuous can be confusing.

This study suggests that there is need for explicit usage and discussion of the terms continuous and discrete in the context of fractions wholes, something that appears to be missing from some mathematics textbooks for elementary school teachers accessed by the researcher (e.g. Billstein, Libeskind, & Lott, 2010; Freitag, 2014; Sonnabend, 2010). One hopes this apparent omission, non-emphasis, or inexplicit usage of the terms in the context of fractions is not because it is assumed that the terms are well understood by all prospective teachers.

The fact that a majority of the participants were able to find fractions equivalent to $3/5$ but failed to demonstrate equivalence using a diagram highlights the need to enrich prospective teachers' pedagogical content knowledge (PCK) by providing plenty of opportunities for them to practice and demonstrate instructional strategies. This example also shows that it is possible that one is be able argue equivalence of fractions using text, for example, $3/5=(3\times 3)/(5\times 3)=9/15$, but fail to correctly argue equivalence using a different representation. Attention should therefore be paid to explanations, illustrations, and representations students use on assessments and in the classroom and connections between different representations of fractions should be explored.

Although some of the participants divided by a number other than 1 in the process of finding equivalent fractions (see Figure 4), the use of the equal sign in the algebraic expressions in Figure 4 is mathematically incorrect since for example $450/720 \div 10 = 45/720 \neq 5/8$. The argument presented to demonstrate $420/720=5/8$ is therefore flawed. Interpreting division by 10 and by 9 as is Figure 4 can be problematic if one has to find say 20% of 10 which involves the calculation $20/100 \times 10$. When done following the example in Figure 4 one incorrectly gets $200/1000$. Explaining how to find equivalent fractions as in Figure 4 may re-enforce or lead to a misconception that dividing or multiplying a fraction by a whole number not equal to 1 gives an equivalent fraction. Meanings ascribed to representations should therefore be probed and most importantly assessments should move beyond manipulation of symbols. This study supports the call by Huang, Liu, and Lin (2009) to promote teacher effectiveness by enriching prospective teachers' knowledge so that it is conceptual and not merely procedural.

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A FRAMEWORK FOR REVISING THE MATHEMATICS TEACHING EFFICACY BELIEFS INSTRUMENT

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The Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) is currently the only instrument in use that measures pre-service teachers' efficacy to teach elementary mathematics (Ward, 2009). Current research suggests that revisions should be made to the MTEBI. The present paper provides the theoretical framework utilized to revise the MTEBI.

The purpose of the present paper is to provide the rationale used for revising the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). According to Ward (2009), the MTEBI is the only instrument available to measure self-efficacy in teaching of elementary mathematics. The MTEBI has been used essentially unchanged since its development. However, recent developments in the field of self-efficacy instrument development, in addition to studies involving the MTEBI, suggest that revision to the instrument is needed (Bandura, 2006; Kieftenbeld, Natesan, & Eddy, 2011; Ward, 2009). The present paper will (1) briefly outline the theoretical framework behind the self-efficacy construct, (2) describe the development of the MTEBI and the studies that prompted the revision of the instrument, (3) detail how the MTEBI was revised, and (4) discuss the next steps in the process of revising the instrument.

Theoretical Framework

There are two theoretical constructs cited in the literature as foundational to the development of the concept of teacher efficacy: (1) the locus of control construct (Rotter, 1966), and (2) the self-efficacy construct (Bandura, 1977b, 1986, 1997). Of the two constructs, Bandura's (1977b) self-efficacy construct, which has its theoretical foundation in social cognitive theory (Bandura, 1971, 1977a, 1986) is the more influential strand of research in teacher efficacy. Bandura (1977b) posited that self-efficacy is comprised of two components: response-outcome expectancies and efficacy expectations. These two components of self-efficacy have served as the focal point of a majority of teacher efficacy research.

In his initial description of the self-efficacy construct, Bandura (1977b) provided the following definitions for the two components of efficacy:

> An *outcome expectancy* is defined as a person's estimate that a given behavior will lead to certain outcomes. An *efficacy expectation* is the conviction that one can successfully

> execute the behavior required to produce the outcomes. Outcome and efficacy expectations are differentiated, because an individual can believe that a particular course of action will produce certain outcomes, but if they entertain serious doubts about whether they can perform the necessary activities such information does not influence their behavior (p.193, italics added).

Bandura noted that it is important to distinguish the differences between efficacy expectations and outcome expectancies.

Bandura (1986) further elaborated on the relationship between efficacy expectations and outcome expectations. He stated that "perceived self-efficacy is a judgment of one's capability to accomplish a certain level of performance; whereas an outcome expectation is a judgment of the likely consequence such behavior will produce" (Bandura, 1986, p. 391). To use an example from the classroom, an individual's belief that he can effectively use manipulatives to teach mathematics is an efficacy judgment, whereas the anticipated impact on student achievement is an outcome expectancy for using the manipulatives effectively.

Bandura (1986) explained two main ways that outcomes can be misconstrued. First, he specified that an outcome is the result of an action, not an action itself. Second, Bandura (1986) pointed out that effective techniques have at times been misinterpreted as outcome expectancies (Maddux, Sherer, & Roger, 1982; Manning & Wright, 1983). Bandura (1986) emphasized "Means are not results. An efficacious technique is a means for producing outcomes, but it is not itself an outcome expectation" (p. 392). Bandura (1986) also discussed how outcome expectations are dependent upon judgments about performance efficacy. Outcomes are not detached from actions; instead, outcomes arise from actions. He stated that because outcome expectancies are dependent upon efficacy judgments, outcome expectancies may add little to the prediction of behavior. He cited several studies that indicate perceived self-efficacy is a much better predictor of behavior than outcome expectancies (Barling & Abel, 1983; Barling & Beattie, 1983; Godding & Glascow, 1985; Lee, 1984a, 1984b; Manning & Wright, 1983; Williams & Watson, 1985).

Development of the Mathematics Teaching Efficacy Beliefs Instrument

The Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) (Enochs, Smith, & Huinker, 2000) was developed to measure pre-service teacher efficacy in mathematics. The 21-item instrument contains two subscales: (1) the Personal Mathematics Teaching Efficacy (PMTE) scale, and (2) the Mathematics Teaching Outcome Expectancy (MTOE) scale. These two subscales are based on Bandura's (1977b) concepts of self-efficacy and outcome expectancy. This instrument has its roots in the Science Teaching Efficacy Beliefs Instrument, Form B (Enochs & Riggs, 1990), commonly referred to as the STEBI-B in the literature. The STEBI-B contains 23 items and has its direct origin to the Science Teaching Efficacy Beliefs Instrument (Riggs, 1988). This instrument is now commonly referred to as the STEBI-A. The STEBI-A was developed by Riggs (1988) to measure the efficacy of in-service elementary teachers. The MTEBI maintains the two subscale structure of self-efficacy and outcome expectancy found in the STEBI-B. In most of the MTEBI items, the term *mathematics* replaces the term *science*. In addition, three items were rephrased to reflect the nuances between using hands-on activities in the science classroom and the mathematics classroom. For interested readers, Ward (2009) provides a detailed description of the evolution of the MTEBI.

Related Literature

There are four strands of research that provide support for the revision to the MTEBI proposed in the present study. First, the literature is replete with studies that stress the importance of measuring teacher efficacy at the appropriate level of content specificity (Bandura, 1997, 2006; Pajares, 1996; Tschannen-Moran & Woolfolk-Hoy, 2001; Usher & Pajares, 2008). The present form of the MTEBI addresses elementary mathematics teacher efficacy, but it does not assess efficacy in the content strands which make up elementary mathematics (numbers and operations, algebra, measurement, geometry, and data analysis and probability) (NCTM, 2000). Second, Bandura (2006) presents guidelines for constructing self-efficacy scales. Three of Bandura's recommendations (content validity, domain specification, and type of scale) are not present in the original version of the MTEBI. Third, several studies suggest that reliability of the MTEBI may not be as high as presently assumed (Guskey & Passaro, 1994; Henson, 2002; Kieftenbeld, Natesan, & Eddy, 2011; Roberts & Henson, 2002; Ward, 2009; Woolfolk & Hoy, 1990). Kieftenbeld and colleagues (2011) suggested that the scale, wording, and placement of the items needed revision, while the other researchers noted specific issues with the MTOE items on the scale. Finally, researchers indicate the need for an instrument that can measure changes in teacher efficacy over time (Bleicher, 2004; Henson, 2002). The present form of the MTEBI is designed for use with pre-service teachers and is not useable with in-service teachers without alteration. Details on how these concerns were addressed relating to the measure of self-efficacy in the revised MTEBI are presented in the methodology section below.

Methodology

The first issue addressed in the revised MTEBI is the level of specificity. The original instrument was designed to measure pre-service teacher self-efficacy in teaching elementary mathematics (subject level). As such, all items in the scale refer to teaching mathematics at the general level. For example, item 5 states: *I know how to teach mathematics concepts effectively*. While this does provide a level of specificity not present in general teaching efficacy scales such as the Teacher Efficacy Scale (TES) (Gibson & Dembo, 1984), it does not address teacher efficacy within the various content strands within mathematics. To address this concern, the format of the instrument was revised so that respondents are asked to answer each question for each of the content strands as outlined by the National Council for Teachers of Mathematics (NCTM, 2000). These content strands are (1) Numbers and Operations, (2) Algebra, (3) Geometry, (4) Measurement, and (5) Data Analysis and Probability. As a result, the format of the instrument was changed to allow respondents to answer each question for each content strand (Figure 1). In addition, since this instrument can be used with pre-service teachers who may not be familiar with all of the content within a strand, the definition of each content strand as defined by the *Principles and Standards for School Mathematics* (NCTM, 2000) is provided as part of the directions for completing the instrument.

Figure 1. Example of modified item

I know how to teach mathematics concepts effectively. **(0 = Cannot do at all, 100 = Highly certain can do)**

Numbers and Operations	0	10	20	30	40	50	60	70	80	90	100
Algebra	0	10	20	30	40	50	60	70	80	90	100
Geometry	0	10	20	30	40	50	60	70	80	90	100
Measurement	0	10	20	30	40	50	60	70	80	90	100
Data Analysis and Probability	0	10	20	30	40	50	60	70	80	90	100

Next, the instrument was revised according to the guidelines detailed by Bandura (2006). First, the issue of the rating scale was addressed. The original instrument provides five options for answering each question: *Strongly Disagree*, *Disagree*, *Uncertain*, *Agree*, and *Strongly Agree*. However, according to Bandura (2006), “efficacy scales are unipolar, ranging from 0 to a maximum strength;” thus, they measure a person’s ability to do a certain task along the continuum from “cannot do at all” to “highly certain can do” (p. 312). He suggests a scale ranging from 0-100 with descriptions of three points on the scale (0-Cannot do at all, 50-

Moderately certain can do, 100- Highly certain can do) or a simpler version using intervals of 10. The simple version with the endpoint descriptors was adopted. This effectively changes the scale from a Likert-type scale with interval data to a ratio scale, since there is a true zero value (cannot do at all) (Hinkle, Wiresma, & Jurs, 2003) (Figure 1). In addition, since this type of response system might be unfamiliar to respondents, a practice rating scale was included at the beginning of the instrument. A practice scale similar to the example provided by Bandura (2006, p. 320) was used. Second, in line with the recommendations of researchers (Guskey & Passaro, 1994; Henson, 2002; Kieftenbeld, Natesan, & Eddy, 2011; Roberts & Henson, 2002; Ward, 2009; Woolfolk & Hoy, 1990), a revision of all of the MTOE items using the term "I" instead of "Teacher" was implemented. This change means that the outcome expectation items should measure outcome expectancy as hypothesized by Bandura (1977b). Finally, additional issues with wording of the items were addressed. Since the original instrument was developed for use with pre-service teachers, some of the items reflect a future orientation ("*I will continually find better ways to teach mathematics*"). Bandura (2006) states that self-efficacy has to do with perceived capability and that "will" is a statement of future intent while "can" is a capability judgment. To address this issue, all items were reworded to reflect an orientation to current abilities as opposed to future intentions. Another issue related to the wording of items is the fact that eight items on the original instrument are negatively worded. The adoption of the 0-100 rating scale made these items problematic because judgments of incapability do not correspond with this scale. Examples of these changes are provided in Table 1 below.

Table 1

Examples of item revisions

Subscale	Issue Addressed	Question Wording	
MTOE	"Teacher"	Original	When a student does better than usual in mathematics, it is often because **the teacher** exerted a little extra effort.
		Revised	**When a student does better than usual in mathematics, it is often because I exerted extra effort.**
PMTE	Future Orientation	Original	I **will** continually find better ways to teach mathematics.
		Revised	**I continually seek better ways to teach mathematics.**

PMTE	Negatively Worded	Original	Even if I try very hard, I **will not teach** mathematics as well as I will most subjects.
		Revised	**I can teach mathematics as well as I teach most subjects.**

After revision of the wording of the items, attention was turned to their placement within the instrument. Kieftenbeld and colleagues (2011) found that several of the items exhibited local dependence with other items in the scale. Some of the local dependence was attributed to the fact that items on the same subscale were placed in close proximity to each other. In other cases, the local dependence was attributed to similarity of content among the items. To address the issue of proximity, PMTE and MTOE items were alternated throughout the scale (two PMTE items followed by an MTOE item). All of the MTOE items were questions that related teacher actions/efforts to student achievement, so it was difficult to do a great deal to avoid the potential for local dependence due to proximity. However, the rewording of the questions did result in the elimination of three MTOE items due to redundancy. In the case of the PMTE items, the content of the questions were analyzed, and it was determined that there were four main types of questions being asked: ability to teach mathematics concepts effectively, use of activities/manipulatives/technology, response to students' questions, and teacher content knowledge. Based on this analysis, similar items were spaced as far apart in the instrument as possible.

The result is a 20 item ratio scale with two subscales. There are 13 items that measure personal mathematics teaching efficacy (PMTE) and 7 items that measure mathematics teaching outcome expectations (MTOE). While it may appear problematic that there are almost twice as many PMTE items as there are MTOE items, as the literature review indicates, researchers have found that self-efficacy (PMTE items) is a better predictor of behavior than outcome expectations (MTOE items) (Bandura, 1986; Barling & Abel, 1983; Barling & Beattie, 1983; Godding & Glascow, 1985; Lee, 1984a, 1984b; Manning & Wright, 1983; Williams & Watson, 1985).

Future Studies

A study to establish the validity and reliability of the revised MTEBI is currently being conducted. The validation study is being carried out during the 2013-2014 academic year. Pre-service teachers in all stages of preparation (from program entry to student teaching) are included in the study. Once the present study is complete, any necessary revisions to the instrument will be made, and a study with in-service elementary math teachers will also be conducted. The result will be a single instrument that can be used with pre-service and in-service teachers. This will provide researchers with an instrument to use in longitudinal studies to assess changes in efficacy over the course of teachers' careers. As research evolves, the findings will be presented at future conferences of the Research Council on Mathematics Learning, at other appropriate conferences, and through peer-reviewed journal articles.

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SAGE AND THYME: CASES OF TEACHER AFFECTIVE DISPOSITION THROUGH THE LENS OF REFLECTIVE TRANSPHENOMENALITY

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Utilizing survey/case study research, middle school mathematics teachers' self-positioning is challenged by positioning-by-others (student) revealing simultaneous transphenomenality as a manifestation of complexity of the main construct of the study – teacher affective disposition. Simultaneity of transphenomenality that reflects "events or phenomena that exist or operate at the same time" (Davis, 2005, p.14) was recognizable in the dynamic and multifaceted nature of disposition which contributed to emergent and shifting mathematical disposition. Fluidity in teacher positioning, measured by multiple instruments, toward mathematics teaching and learning was representative of metamorphoses between teacher-as-engineer and teacher-as-technician and resultant student disposition toward mathematics.

Purpose of the Study

The purpose of the exploratory study was to ascertain if there was a pattern of transphenomenal simultaneity, or events or phenomena that exist or operate at the same time (Davis, 2005), between teacher and student disposition that contributed to the phenomenological conflict of teacher self-positioning and positioning-by-others toward mathematics, mathematics teaching and learning. In this context, self-positioning occurs when responses to mathematics discourses and cultures are situated in reaction to, and in order to navigate academic content, settings, and interactions. Positioning-by-others is characterized by occupying more than one level of positioning at a time (Davis, 2005). The intent was to provide a view of self-reported and observed middle school teacher and student experiences in mathematics, based on the analysis of affective dispositional characteristics within the context of its complexity.

There is evidence (Beyers, 2011) that documents a relationship between teacher and student disposition, but to what extent and encompassing what characteristics and factors has not been sufficiently substantiated. Beyers' (2011) synthesis of the literature identified two key impacts of disposition on learning: (1) "…teachers play an essential role in shaping students dispositions with respect to mathematics", and (2) "students dispositions with respect to mathematics affect student learning by means of opportunities to learn" (p. 70). For purposes of this study, positioning was defined as habitual inclination formed by teacher and student in response to and in order to navigate through academic content, settings, and interactions. The intent of this study was to investigate and delineate the nature of the positioning, as evidence of transphenomenality in self-positioning and positioning-by-others.

The research questions were 1) *Within the complexity framework, what are teacher and student affective disposition characteristics which contribute to a phenomenological conflict between teacher self- positioning and positioning-by-others?*; and 2) *How, and to what extent, is transphenomenal simultaneity in teacher positioning reflected by student positioning?*

Theoretical Framework

The components of the theoretical and contextual frameworks were grounded in Positioning (Harré, 2011) and Complexity Theories (Davis, 2005; Davis, 2008) guided by Social and Sociotransformative Constructivism (Bogdan & Biklen, 2010; Rodríguez, 2005). Education, by the very nature of the concept, is complex. The complexity of education is rooted in the nature of discourse, multiplicity and simultaneity. Osberg (2005) suggests that education is dynamic, with intertwined variables and "we educate in what might be called a 'space of emergence'…", teachers and students participate in the educational process "…from a position of extreme flexibility and responsiveness to the moment or space we are in" (p.82). To conduct educational research from a limited narrow perspective without regard for transphenomenal simultaneity or the connections between the *knower, knowledge* and *learner* is not accounting for "how discourses intersect, overlap, and interlace" with the phenomena (Davis & Phelps, 2005, p.3).

The positioning of the teacher may evolve in one of two ways: as teacher-engineer or as teacher-technician (Tchoshanov, 2011). Attributed to Bourdieu (1991), Uljens (1997) situated positioning as teacher-technician or teacher-engineer in reflective theory of didactics, describing it as a method of ascertaining "…how instructional processes in the institutionalized school may be experienced" (p.v) in relation to the teacher positioning. Critical to this study were the variances in self-positioning and positioning-by-others.

Sociotransformative Constructivism (sTc) acknowledges the role of power structures, locations, and institutional codes in the development of affective dispositions and interpersonal positioning (Rodriguéz, 2005). Within sTc, transphenomenal simultaneity exists and positioning is impacted by power (agency) relationships and institutional codes (institutional, historical, and social) and locations (social, ideological and academic). Positioning Theory "… pictures a dynamic stability between actors' positions, the social force of what they say and do, and the storylines that are instantiated in the sayings and doings of each episode" (van Langenhove &

Harré, 1999, p.10). Encompassing diversity of purpose, Positioning Theory allowed for framing research focus and concepts within social discourse/meaning-making analysis utilizing constructs of affective disposition to ascertain self-positioning and positioning-by-others.

Method of Inquiry

Data was collected from a survey instrument which employed open-ended questions and metaphorical prompts (Stage One), case study Likert-type ratings (Stage Two), case study interview prompts (Stage Three), case study classroom observation protocol and field notes (Stage Four); these data were the foci of coding and meaning analyses of self-reported and observed participant responses. The multi-stage mixed methods approach produced a shift from quantitative methods in Stages One and Two to a heavily emphasized concentration of analysis grounded in qualitative examination of data in Stages Three and Four.

The study structure utilized affective domain characteristics of disposition as guiding descriptors of teacher and student positioning. A synthesis of descriptions and definitions of affective domain characteristics were developed as the categories of nature of mathematics, worthwhileness, usefulness, sensibleness, self-concept, attitude, and anxiety and were used to facilitate and guide the coding and determination of teacher and student self-positioning.

Linguistic deconstruction, meaning-making, and discourse analysis (Kvale & Brinkmann, 2009) were the main tools of coding. Independent 'expert' raters determined levels and intensity of dispositional inclinations from survey responses. Inter-rater reliability was achieved through use of Pearson's *r*, Delphi Method of Consensus and Cohen-Fleiss Kappa, accounting for multiple raters. Open coding of self-reported and observed data included frequency and meaning coding and analyses were performed according to operationally defined clusters and identification of themes or nodes.

Participants

Participants were located in a large school district in a border region of the U.S. and Mexico. In Stage One, teacher and linked students yielded a sample of 22 teacher surveys and 458 corresponding student surveys. For Stages Two – Four, two teacher case studies were identified with criteria-based sampling of self-positioning as teacher-engineer and teacher-technician. Criteria-based sampling identifying polar opposite case studies was used to substantiate the presence of transphenomenal simultaneity reflective in diverse self-positionings.

Demographically, the two teacher case studies, Sage and Thyme (pseudonyms) were eighth and sixth grade female mathematics teachers, Hispanic and Anglo, and possessed a bachelors and master's degree, respectively. Each teacher had accumulated in excess of twenty years of teaching experience. Three linked students per teacher were chosen by indicated positive, neutral, or negative self-positioning toward mathematics, mathematics teaching and learning and willingness to participate in the interview stage. Of the six students, 50% were male and 50% were female, 17% were African American, 33% Hispanic, 50% Anglo, and two students reported as bi-lingual English-Spanish, while all others were mono-lingual English speakers.

Data Collection and Analysis

Primary attention was given to the interconnectedness between teacher and student affective dispositions and positioning and how the self-positioning of teachers was challenged due to positioning-by-others. Analysis and synthesis of the data required a mixed methods approach to holistically reflect the authentic responses of the participants within Positioning, sTc and Complexity Theory frameworks. Data were collected in four stages for the purposeful sample and case studies.

In Stage One, survey open-ended and metaphorical prompt responses were coded according to levels of defined affective dispositional constructs and ranked by multiple expert raters within an operationally designed scale (1-highly nonproductive to 5-highly productive). In Stage Two, Likert-type rating and narratives were numerically ranked, analyzed through linguistic deconstruction and comparatively analyzed with Stage One results. In Stages Three and Four, interview transcripts, interviewer observations, classroom observations and field notes were categorized in clusters of affective dispositional characteristics with meaning coding and frequency of response/observation measurements utilized to identify themes or nodes of affective disposition reflective of self-positioning as teacher-engineer and teacher-technician. State assessment and demographic data were also collected to provide additional quantitative data (confirming and/or disconfirming) for triangulation.

Results and Discussion

The overarching finding in Stage One demonstrated generally an overall positively inclined affective disposition mean rating for 22 teachers (*3.38*) and a slightly lower overall mean rating (*3.24*) for the 458 linked students. This finding indicated a contradiction to the literature review finding of prevalent negative dispositions toward mathematics (Rock & Shaw, 2000; Picker & Berry, 2000). Rock and Shaw (2000) "…concluded that students believe that a mathematician's job is to do the mathematics that no one else wants to do" (p.553) or an image of mathematicians functioning beyond the student cognitive levels. However, further exploration of observed data for identified case studies was needed to substantiate positively inclined results found.

Stage Two results of the Likert-type ratings for Sage and Thyme continued to reflect inflated and deflated self-reporting of disposition toward mathematics, mathematics teaching and learning. Sage rated herself in all areas as '*above average*': *"I share my knowledge with students. I leave it mostly to them to learn from my experience"*. Sage's linked students self-rated disposition from below average to above average, each with unique narratives*: "Average because I'm not outstanding but I get how it works; below average because I'm not very good at explaining"*, *"I'm a fast learner when it comes to math, but not the best",* and *"Sometimes I struggle when they're explaining to me at first, then I get focus and I understand it".* Thyme rated herself as '*average to above average*' but in the narrative expressed doubt of her content mastery: "*I am not afraid to learn alongside with my students and this encourages and facilitates a positive learning environment*". Her students self-rated as average to above average including revealing narratives of positioning-by-others: *"Average because my teacher doesn't explain stuff very well"*, *"Above Average, my teacher hasn't done anything wrong"* and *"Average, if my teacher explains things to me, I understand it better"*. As in Stage One, this finding was solely based on self-reported data and required further exploration through additional data collection.

"Complexity theory proposes that any minute change in any dynamic system has a generative impact on a multiplicity of inter-related locations and relationships" (Fels, 2004, p.77). It was surmised in Stages Three and Four that teacher and student core dispositions dynamically manifested in varied self-positioning stances and subsequent positioning-by-others from diametrically opposed perspectives. For example in Stages One to Three, Sage positioned herself as a teacher-engineer or *Sage on the Stage*: *"I'm sharing my math knowledge with students. I have lots of math knowledge…".* However, in Stage Four, Sage was observed demonstrating teacher-technician attributes as evidenced in 97% of all semantic references which indicated positioning as a teacher-technician in themes or node categorizations. Field notes from the classroom observation quoted Sage: *"Turn to page 23 and copy what I have here. Turn to page 12 of your STAAR interactive notebooks. Page 17, my bad." Goes through True/False answers.*

Continues reading answers "Now Lesson 15, 1-10". The description of basic, step-by-step procedures is representative of Sage's positioning as a teacher-technician.

Fluid shifting from self-reported teacher-engineer to observed teacher-technician exemplified transphenomenality in response to student positioning or positioning-by-others. Of the three students linked to Sage, all three students experienced the same teacher positioning toward teaching and learning; nonetheless students exhibited distinct, unique core dispositional inclinations – positive, neutral, or negative. Student Three (negative) stated in the interview "*When you're just doing work all day you just feel like I want to stop this and I want to go home. But if there was like some fun things. Like maybe working or sitting next to a partner and having a partner and that would like work better, I guess. Because you don't feel like you're forced to be quiet and forced to do it – you want to do it.*" The academic year 2011-2012 student state assessment data further supported Sage's positioning-by-others as a teacher- technician with overall student raw percent scores in the quartile mid 50% (lowest among participating teachers).

In contrary, Thyme in Stages Three and Four of the study positioned herself as a teacher-technician. However, observationally, and as supported by state assessment data, she was positioned-by-others as a teacher-engineer. Thyme was observed to be thoughtful and more individualized in her teaching practice exhibiting teacher-engineer characteristics. Field notes stated *"Students interacted around the topic. Learning was enriched by conjecture, investigation and analysis. Instruction was related to something relevant to the students. She asked for explanations and justifications"*. Additionally, academic year 2011-2012 linked student state assessment scores were reported in the upper quartile of participating teachers or in the mid-60% range. Student comments indicating Thyme's positioning-by-others as a teacher-engineer included: *"I kind of liked solving it because you know it was a little bit harder than what I was used to like the adding and subtracting, but it was pretty easy once I got the hang of it."*

Findings from Stages Three and Four demonstrated that simultaneity of transphenomenality through "events or phenomena that exist or operate at the same time" (Davis, 2005, p.14) was recognizable in the dynamic, multifaceted nature of self-positioning and positioning-by-others which contributed to emergent and shifting mathematical disposition.

Conclusions

In Stage One and within the framework of Complexity Theory, inflated/deflated self-reported narrative of disposition and self-positioning encouraged embracing "…disorders (Alhadeff

Jones, 2012b) rather than systematically looking for order." (Alhadeff-Jones, 2013, p.42).

Stage Two self-reported rating of disposition via Likert-type scale provided a limited, yet structured rating of disposition. Even with interpretation of narrative, the Likert-rating provided little additional evidence in contradiction to the observation of inflation and deflation. Case study participants rated themselves on the Likert-type scale similarly to self-reported ratings identified in the survey research component. These findings further support that the complexity of self-reported storylines or narratives of affective disposition cannot be compartmentalized into pre-determined levels or rankings. The linear and straightforward nature of quantitative data in Stages One and Two was challenged by holistic qualitative analysis in Stages Three and Four to embrace disorder and simultaneous multiplicity of transphenomenal characteristics of affective disposition.

In Stage Three, through meaning interpretation, it was found that trajectory of student disposition, as reflection of teacher disposition, is dependent on multiple factors within the learning environment and is unique to each individual. Although students may simultaneously be exposed to the same teaching method and materials, they may have very different trajectories or outcomes in their positioning.

Stage Four analyses found that in the two case studies, Sage and Thyme, self-positioning and positioning-by-others demonstrated simultaneity of transphenomenal affective characteristics of disposition. Sage self-reported inflated characteristics of a teacher-engineer and Thyme self-reported deflated characteristics of a teacher-technician. In subsequent analysis the phenomenon of student reflection of teacher practice demonstrated converse positioning of teachers – Sage as a teacher-technician and Thyme as a teacher-engineer. This is not to say that self-reported disposition was false, but rather due to the complexity of the construct and the influence of extraneous factors, *action spoke louder than words* and was considered to be the authentic depiction of teacher disposition toward mathematics, mathematics teaching and learning.

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ATTITUDE ADJUSTMENT IN INTRODUCTORY STATISTICS

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Many students that enroll in introductory statistics courses do not have positive attitudes about the subject. A recent wide-ranging study showed that student attitudes do not tend to improve after completing an introductory statistics course. However, there is a need for more studies about attitudes in introductory statistics courses that utilize reform teaching methods. In this paper, we present findings about student attitudes towards statistics in both a teacher-centered lecture-based class and in a student-centered collaborative-learning class taught by the same instructor. The differences in attitudes that emerged are discussed.

Undeniably, making informed decisions in an ever-increasing global society is greatly enhanced by a strong understanding of how to deal with numbers and data. Statistical literacy and statistical reasoning are important components to understanding the economy, Gross Domestic Product, inflation, government decision-making, and much more. Therefore, as a part of preparing university students for independent decision-making, more and more majors are requiring at least one statistics course prior to graduation. Nevertheless, even those students who complete such a course do not necessarily emerge from these statistics classes with a solid comprehension of the material presented (American Statistical Association, 2005). Instructors and researchers alike tend to focus on what should be taught and how it should be delivered to address this issue of retention. However, recent research in statistics education has shifted focus to include student attitudes when examining student comprehension of statistics (Bond, Perkins, & Ramirez, 2012). The idea is that student attitudes can impact student openness to learning statistics. The purpose of this study is to examine student attitudes at the beginning and end of two sections of an introductory university statistics course where different instructional methods were utilized. The authors aim to understand if learning statistics, with the same teacher, in a teacher-centered versus student-centered course has an effect on students' attitudes towards statistics; if so, in what ways? We first present some background literature of student attitudes towards statistics, followed by the methodology of this study, then the results of our work, and finally a discussion of those results.

Background Literature

In the early 1990s, Candace Schau, a university statistics professor, began to notice that attitudes were an important part of her students' abilities to learn statistics (Schau, Millar, & Petocz, 2012). Along with the aid of some graduate students, she developed the Survey of Attitudes Towards Statistics (SATS-28), currently the SATS-36 (Schau, 1992, 2003). Essentially, in order to help students embrace the study of statistics, they needed to develop positive attitudes towards the subject. The SATS created an avenue for understanding student attitudes prior to engaging in a statistics course and for examining what happens to those mindsets after taking a statistics course. Since the development of the instrument, statistics educators across the world have been utilizing this tool to study attitudes of students as well as statistics teachers. The reliability and validity of the instrument has been critically analyzed by several researchers (e.g., Chiesi & Primi, 2010; Vanhoof, et. al, 2011).

With the notable increase in popularity of the instrument, Schau and Emmioglu (2012) collected data for SATS-36 from approximately 2200 post-secondary students across the United States. Data were collected when students entered and left introductory statistics courses. Results of the study showed that in general, attitudes towards statistics either decreased or did not change significantly after the completion of an introductory course. However, Bond, Perkins, and Ramirez (2012) have supported Gal et. al's (1997) claim that more research is needed in assessing attitudes in classrooms that utilize reform teaching methods. Therefore, the authors of this paper compared and contrasted the attitudes of students enrolled in two sections of introductory statistics taught in two different ways by the same instructor.

Methodology

This study was a classic quasi-experimental design with a control and experimental group. This research compared the beliefs of students from one section of an introductory statistics course that was taught using a traditional lecture-based approach and a second section taught using a student-centered approach. This particular 100-level course is one option that satisfies the general education mathematics requirement at a regional university in the Southeastern United States. The breakdown of majors between the two sections varied a little. Just over 50% of the students in the lecture-based section were STEM majors. In contrast, the non-lecture section consisted of about 42% that were STEM majors. Furthermore, the lecture-based course was made up of approximately 70% females and 30% males; whereas, the nontraditional section consisted of about 58% female and 42% males. Table 1 contains the breakdown of academic rank for each section of this course.

Table 1. *Academic Rank by Section*

	Lecture	**Non-Lecture**
Freshman	3	7
Sophomore	9	8
Junior	12	6
Senior	3	2
Graduate	0	1

The same instructor taught both sections of the course, which met three times per week, with the lecture section meeting at 11:30 AM and the non-lecture section meeting at 1:50 PM. The lecture section was traditional in that the instructor lectured while students took notes and then completed similar problems on their own (both in class and via homework). In the non-lecture section, students were expected to encounter the material on their own prior to coming to class (via a guided set of questions). This was followed by small-group and whole-class discussions in class. For detailed information regarding how the two sections of this course were taught, please refer to Autin, Bateiha, and Marchionda (2013). While the instructor had over eight years of experience teaching statistics at that time, she did not have prior experience with teaching a course in a non-traditional manner. Therefore, she worked closely with two mathematics education colleagues to design and implement this method of teaching.

The Survey of Attitudes Towards Statistics (SATS-36; Schau, 2003) was used to assess student attitudes before and after having taken this introductory statistics course. The SATS-36 consists of 36 questions (worded both positively and negatively) that make up six components designed to measure attitudes related to statistics. The components are Affect, Cognitive Competence, Value, Difficulty, Interest, and Effort (see Table 2). These are measured using a standard 7-point Likert scale where 1 corresponds to “strongly disagree,” 4 is “neither disagree nor agree,” and 7 corresponds to “strongly agree.”

Table 2. *SATS-36 Attitude Component Descriptions*

Component	# of items	Designed to Measure
Affect	6	Students' feelings concerning statistics
Cognitive Competence	6	Students' attitudes about their intellectual knowledge and skills when applied to statistics
Value	9	Students' attitudes about the usefulness, relevance, and worth of statistics in personal and professional life
Difficulty	7	Students' attitudes about the difficulty of statistics as a subject
Interest	4	Students' level of individual interest in statistics
Effort	4	Amount of work the student expends to learn statistics

(Schau, 2003)

Students completed the SATS-36 pretest survey on the second day of class and the posttest survey on the last day of class, prior to the final exam day. Participation was voluntary, and student responses were recorded in a spreadsheet after the end of the semester. The questions that were negatively worded were reverse coded. Each student's mean score for each component was analyzed (Schau, 2003). If a student failed to answer a question for a particular component on the pre or posttest, his/her responses were disregarded for that component (see Table 3 for sample sizes). Higher scores for any component indicated a more positive attitude. For instance, a higher score in Difficulty on the pretest and posttest indicated that students believed that statistics is not a difficult subject. A higher score for Effort on the pretest indicated that students planned to work hard in the class, while a higher score on the posttest indicated that they believed they did work hard in the course. All analyses reported in the results section were performed using R (R Core Team, 2012).

Results

Students' mean pretest scores, as well as the variability in these means, differed across components, as shown in the boxplots in Figure 1. Students tended to have very positive attitudes (and the least amount of variability) about their Effort at the start of the semester, while they started the semester with the poorest attitude about Difficulty. Although some differences can be seen in the lecture and non-lecture sections, the boxplots do not indicate whether or not these differences are significant. Since the sample sizes were small and the data was collected using a Likert scale, the nonparametric two-tailed Mann-Whitney test was used to analyze this data for each component. All p-values (see Table 3) were found to be non-significant; this meant that there was no significant difference between the two sections in the student attitudes about statistics at the start of the semester.

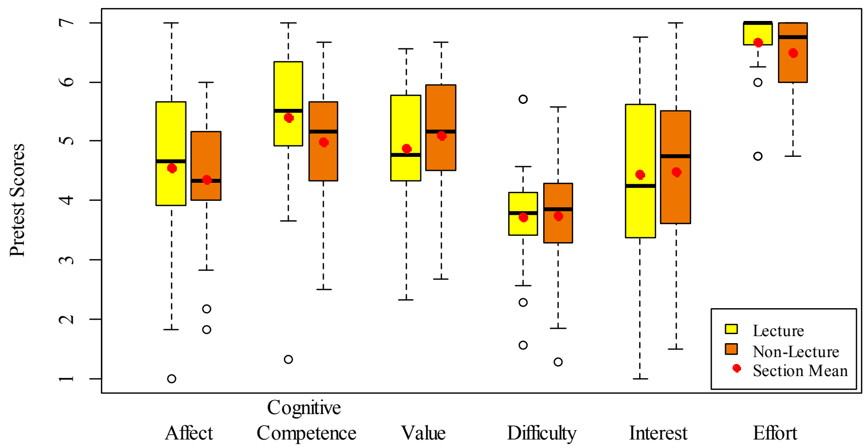

Figure 1. Boxplots of student mean pretest scores by section and attitude component.

Mean posttest scores for the students are presented in Figure 2. As was the case of the pretest scores, students tended to have the most positive attitudes in the Effort component and the poorest attitudes about Difficulty. There tended to be more variability in mean posttest scores than mean pretest scores, as indicated by the interquartile ranges (IQR). In fact, most of the interquartile ranges are larger for the posttest scores than for the corresponding pretest scores with the following three exceptions out of the twelve: the lecture class's Cognitive Competence IQR and the non-lecture class's Value IQR remained the same, while the non-lecture class's Effort IQR decreased by 0.25. Two-tailed Mann-Whitney tests comparing the posttest scores for the two sections (see Table 3) revealed that the only significant difference was in Effort. Students in the non-lecture class had significantly lower attitudes about the amount of effort they put into the course (attendance, assignment completion, etc.) at the end of the semester than those in the lecture class.

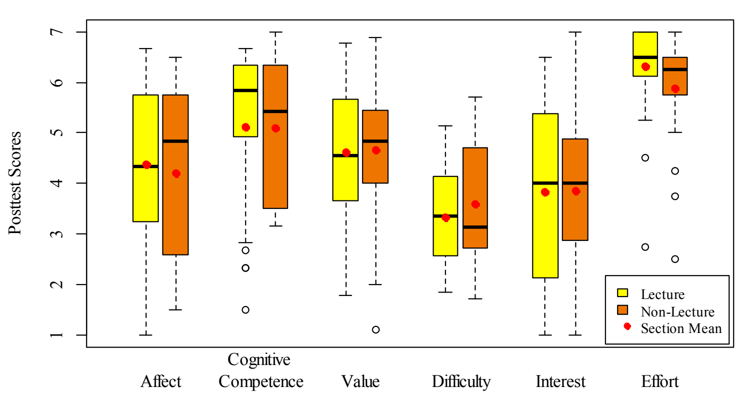

Figure 2. Boxplots of student mean posttest scores by section and attitude component.

Figure 3 contains boxplots for the students' mean changes (posttest – pretest) in score for the six attitude components. More than 50% of non-lecture students had positive changes in Affect and Cognitive Competence, as indicated by the positive medians. This signified that a majority of the non-lecture students indicated a positive shift in their feelings concerning statistics (Affect) as well as about their own knowledge and skills when applied to statistics (Cognitive Competence). A large majority of students (more than 75%) in both sections had negative changes in their Effort attitudes, indicating that they did not actually expend as much effort into this class as they thought they would at the start of the semester. Mann-Whitney tests (see Table 3) revealed that there were not significant differences in the changes in attitude between the two sections for any of the six components.

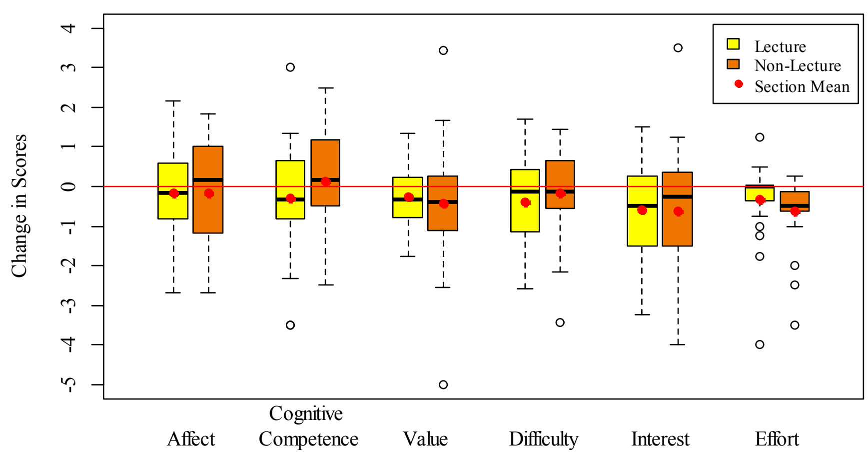

Figure 3. Boxplots of student mean change in scores (posttest – pretest) by section and attitude component. The red reference line at 0 indicates no change in attitude.

Table 3. *Results from Mann-Whitney Tests Comparing the Two Sections*

	Sample Sizes		p-values		
	Lecture	*Non-Lecture*	*Pretest*	*Posttest*	*Change*
Affect	27	23	0.4889	0.7256	0.9456
Cognitive Competence	27	22	0.1007	0.8167	0.3919
Value	25	24	0.5348	0.9203	0.7561
Difficulty	26	23	0.7939	0.5208	0.3608
Interest	27	24	0.9698	0.8947	0.9623
Effort	27	23	0.2830	*0.0289*	0.1103

Significant results indicated in red italics.

Although Figure 3 shows that, overall, most of the changes in attitude tended to be negative (with the two exceptions of Affect and Cognitive Competence in the non-lecture section), to test for significance, the two-tailed Wilcoxon signed rank test was used to compare paired student pretest and posttest mean scores. Table 4 contains the p-values that resulted from each of the twelve tests performed. There was a significant decline in Effort for both sections, with students not meeting the work-ethic expectations they put upon themselves at the start of the semester. Additionally, there was a significant change in Interest for the lecture section; lecture students' level of interest in statistics declined over the course of the semester. To explore this, further analysis regarding this component was conducted. Change in Interest attitude for the lecture students was found to be significantly positively correlated with student classification (Spearman correlation of 0.4192; p-value = 0.0148). Thus, students that were further along in their college career tended to have more positive changes in their interest in statistics than those who were "younger". Additionally, a Mann-Whitney test revealed that the change in Interest is significantly higher for upperclassmen than it is for underclassmen (p-value = 0.0095). For the non-lecture class, no such significant relationship between change in Interest and student classification was found.

Table 4. *p-values from Wilcoxon Signed Rank Test for Change in Attitude*

	Lecture	**Non-Lecture**
Affect	0.5271	0.6589
Cognitive Competence	0.5004	0.7086
Value	0.0884	0.1487
Difficulty	0.1063	0.9224
Interest	*0.0367*	0.1092
Effort	*0.0322*	*0.0006*

Significant results indicated in red italics.

Discussion

For both sections, students' attitudes about the amount of effort they would put into this statistics class decreased significantly from the beginning of the semester to the end. Students started the semester with high expectations of themselves; they seemed to believe that they would attend class regularly, study hard, and complete all assignments. However, at the end of the semester, they seemed to recognize that this did not happen. Regardless, this was still the attitude component in which students overall tended to report having the most positive attitude on the posttest, even though it was not as positive as at the start of the semester. This is consistent with the results that Schau and Emmioglu reported in their wide-ranging study (2012). It is interesting that there was a significantly lower posttest attitude for Effort in the non-lecture section than there was for the lecture section since the non-lecture class was designed in such a way that students were required to put forth more effort. This might be because much of the effort students put forth to understand the material was during in-class group discussion rather than outside of class; additionally, there was more collaboration in making sense of difficult concepts. Therefore, they viewed themselves as putting forth less effort overall than they thought they might.

Although the positive changes seen in a majority of the non-lecture students' Affect and Cognitive Competence were not statistically significant, they are promising. They indicate that there is a possibility that attitudes can be positively affected by learning in a student-centered classroom environment. In a previous publication, the researchers described some limitations to the way the student-centered course was taught (Autin, Bateiha, & Marchionda, 2013). Since this was the instructor's first attempt at teaching in such a way, it was a learning experience; after reflection, the instructor noted that she would teach the course differently in the future, addressing some of the problems that arose in the first go-around. More studies of student-centered statistics courses would need to be investigated to determine if teaching in this way could lead to significant positive changes in Affect and Cognitive Competence.

The significant change in Interest for the lecture class indicates that, unlike students in the non-lecture class, these students lost interest in the subject. This could be an indication that students find a student-centered environment more interesting, which could contribute to positive student motivation. Further analysis indicated that upperclassman in the lecture section seemed to have a positive shift in interest in comparison to younger students. This could be due to a more intrinsic desire to learn the material than what exists for a less mature university audience. However, further questioning is needed to investigate this more.

The increases in variability seen in the posttest scores (Figure 2) indicate that students' attitudes differ more among each other at the end of the semester than they do at the start of the semester. Several factors could have contributed to this result including but not limited to some students losing/gaining enthusiasm for learning statistics after performing poorly/well in the course.

The results of this study were similar to those reported in other research studies, indicating that the students at this university are similar to those at other universities with regard to attitudes towards statistics (e.g., Schau, Millar, & Petocz, 2012). What is important to note is that a comparison was made between two sections of statistics that were taught in drastically different ways. Given the nature of instruction in those two sections and the instructor's reflections that the non-traditional section could be improved upon based on what she learned (Autin, Bateiha, & Marchionda, 2013), further research is needed to determine if teaching in a student-centered, collaborative-learning environment will improve attitudes as they relate to statistics. In addition, further research is needed to investigate why there were significant changes in Interest and Effort.

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USING IPADS TO INFLUENCE INSERVICE TEACHERS' PEDAGOGY IN THE MATHEMATICS CLASSROOM

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For this qualitatively based study, the researchers examined how graduate mathematics instruction using iPads influenced 22 middle school teachers' (Grades 6-8) utilization of technology in their classrooms. The researchers collected data from written reflections and iPad application assignments. Based on analysis of data, the most popular uses of iPads in the classroom included lesson delivery, tutorial aides, and review activities.

As evidenced by society's dependence on electronic media, technology plays an increasingly important role in many people's everyday lives in the United States. U.S. schools are no different, with many districts turning to computers and even iPads to aid in lesson delivery and assessments. According to the National Council of Teachers of Mathematics (2000), "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 11). Technological devices, such as iPads and calculators, help instructors bring mathematics to life through engaging lessons that support standards-based mathematics instruction. Today's students are equipped to investigate mathematical relationships and problem-solving scenarios with more sophisticated technology than in years past. Even though students are seemingly ready for this change in instruction, they may not be given that chance if teachers do not support these changes.

To examine these issues, we investigated the following research questions in two 3-credit hours graduate mathematics education classes for inservice teachers:

1. How does mathematics instruction with iPads influence middle school teachers' uses of technology in mathematics instruction?
2. What challenges do inservice teachers face by implementing iPads in their mathematics classrooms?

Literature Review

A review of the literature supports the use of technology in the classroom and emphasizes the role of the teacher. Information, communications, and technology (ICT) literacy is listed among the 21st century learning skills identified for student success in the new global economy

(Partnership for 21st Century Skills, 2013). However, technology's impact on student achievement is highly dependent upon the availability of resources and the teacher's knowledge and skills (Hew & Brush, 2007). The emphasis on developing teacher knowledge and skills related to technology integration is further evidenced by the inclusion of the 'use of technology' as one of the core teaching standards of The Interstate Teacher Assessment and Support Consortium (CCSSO, 2011).

Even though technology is seen by many as an essential tool to teaching and may be physically present in classrooms, its utilization may be lacking. According to a survey conducted by the National Center for Education Statistics (NCES, 2000), 98% of all secondary teachers have access to at least one computer in the classroom. However, this same survey reports that only 24% of these teachers report they "often" use this technology during instructional time. Specifically, for teachers of mathematics/computer science and science, 99% have access to computers in the classroom everyday though only 21% report they "often" use this technology during instructional time.

In addition to the frequent use of computers, document cameras, and the Internet for research and communication, more recently, mobile devices such as iPads have become increasingly appealing and available for their pedagogical uses. Kearney and Maher (2013) studied the ways in which mobile learning technology (iPads) could be used in the professional learning of pre-service elementary teachers in mathematics. The study focused on the participants' use of iPads (issued to each participant) in professional learning activities (mathematics and pedagogy focused), and organizational, reflection, and communication skills.) The results of the study suggested that though the organizational benefits (planning, note-taking, record-keeping, reflective practices, and collaboration) were evident, participants were in the early stages of contemplating how to use ICT-enabled opportunities to enhance mathematics instruction. For example, participants demonstrated the ability to use the iPads to capture images situating mathematics in a real-world context and the observation of out-of-class mathematics phenomena. Participants shared the images and observations with their peers (one-on-one or small group) in order to stimulate discussion and collaboration. Although the professional development was focused on mathematics education, participants clearly favored the applications (apps) available for communication and productivity.

Murray and Olcese (2011) studied the potential effect of both iPads and iPad apps on teaching and learning in K-12 learning settings. In particular, a representative selection of education apps available through iTunes (approximately 30,000 in 2010) was reviewed. The researchers first organized apps according to the following categories: (a) tutor, (b) explore, (c) tool, or (d) communicate. The apps were then reviewed to evaluate the extent to which they promoted *communication and collaboration*, a learning and innovation skill from the Framework 21st Century Learning (Partnership for 21st Century Skills, 2009). Findings from the study suggested there were few apps that extended what teachers and students are able to do in the classroom. Although apps linking the user to social networks (e.g., Facebook, Twitter) and shared files (e.g., Dropbox, iCloud) allowed users to share information, and online textbooks and virtual manipulative sites provided more economical alternatives, the majority of education apps merely provided a drill and practice platform. Furthermore, the results suggested that the majority of apps were "woefully out of sync with modern theories of learning and skills students will need to compete in the 21st century (Murray & Olcese, 2011, p. 48) as they focused on the presentation of facts rather than engaging the user in creative thought.

In order to successfully integrate technology into their classrooms, teachers must feel comfortable with technology and perceive the integration of technology to be an improvement on what is already being done. Teachers are hesitant to use methods that have not been personally observed and deemed worthy of the time necessary to integrate new ideas or strategies (Buabeng-Andoh, 2012). Ongoing technology training opportunities and modeling, opportunities for teachers to collaborate and access to technology support are ways to enhance teachers' readiness.

This study addressed key factors in the implementation of technology in the classroom. The key factors incorporated into the goals of the mathematics education courses in this study included providing teachers with a) an opportunity to investigate, review, and learn to manage mobile technology, and b) an ongoing support system essential to the successful implementation of technology into middle school mathematics courses and instruction. Also studied was the impact of the mathematics education courses on another key factor, the teachers' attitudes, and beliefs towards technology integration (Buabeng-Andoh, 2012; Mumtaz, 2000).

Methodology

The study participants included 22 middle school (Grades 6-8) teachers from two school districts in the southern United States. These teachers were part of the second year of a federally funded grant focused on middle school geometry content, where inclusion in the program centered on teachers who were (a) alternatively certified with less than 3-5 years of experience teaching, and/or (b) deficient in one or more of the following areas: certification in mathematics, collegiate geometry/mathematics courses, and/or mathematics pedagogy classes.

The teachers took part in two 3-credit hour graduate courses, (a) a 2013 summer course, and (b) a 2013-2014 academic year course. One of the researchers for this study was one of three instructors for the program. The summer course was held over 9 days, 6 days (8:30-12:45 each day) of class instruction and 3 days of attendance at a standards-based instruction mathematics education conference. The academic year included 8 Saturday classes from 8:30-4:00 held over the course of the 2013-2014 academic year. Course instructors emphasized geometry instruction, such as area, surface area, volume, polygons, and solids, using concrete manipulatives during the fall and iPad apps during the summer and fall.

As part of the grant, participants received iPads with the intent that they would integrate the new technology and strategies into their classroom instruction beginning fall 2013. Thus, the summer course focused on the iPad and iPad apps. The instructors of the course researched and selected 28 geometry and pedagogical apps for teachers to download on their iPads. The participants then randomly chose 1-2 of the apps to research and then completed a project about their app(s). This assignment included three components, an iPad reflection about their previous technology knowledge including ways they would incorporate their app(s) in their class, an iPad handout to provide for other participants for reference purposes, and a 35-45 minute presentation about their app(s).

In order to answer the research questions, we collected written reflections and iPad assignments. Reflection data were collected during the summer course, as well as the two September and one October class meetings. Reflections were completed through Edmodo, a free website resource where teachers can create courses so that students can complete various class activities, including assignments, quizzes, and polls. The iPad assignments of importance for this study included reflections in which participants discussed their experience with technology and iPad apps. A discussion of this analysis with pseudonyms given for participants follows.

Findings

Research Question 1

From analyzing the September 2013 and October 2013 reflections, we found only 2 of the 22 teachers had more than a semester's worth of experience with the iPad prior to the grant classes. Of the 8 teachers who had any prior experience with the iPad before the grant, only 3 had utilized the iPad for teaching purposes, such as a mobile whiteboard. The other 5 teachers used the iPad more for email than instructional purposes.

Even though only a few participants had prior experience with the iPad at the start of the study, 15 used the iPads in their classes in fall 2013. From the two September reflections, we found that 10 of the 22 participants had used the iPad in their instruction during their first three weeks of school. The most popular uses were for instruction and tutoring. Five reported using the iPad as an instructional delivery tool, and five used the iPad as a tutorial aide. From the October reflections, we found all 10 participants who used the iPads in September continued to use the iPads in their classrooms in October. Also, an additional five participants utilized the iPads. While one of the additional iPad users implemented the iPad for lesson delivery and another for tutorial purposes, the other three participants used a music-related application, AutoRap that was suggested by Jasmine, a participant, during one of the September meetings. There were a total of six participants that utilized this application, whereby the participants had students create raps to summarize mathematical concepts as a review activity.

One of the most energetic iPad users was Lauren. In an Edmodo reflection, she detailed how the iPad and in particular the productivity application Doceri had been invaluable during the first weeks of school:

> So far, I have used Doceri in my classroom when I was unable to use my Promethean board. I have also had my students create Edmodo accounts, and I used Show Me to create and post tutorial videos that go along with students' take home notes. I plan to continue creating these videos throughout the year…This iPad has been a godsend to me for my first two weeks of school.

Other iPad uses included the following: creating a seating chart (1 participant), taking attendance (1 participant), making pdfs of paperwork (1 participant), developing QR codes for open house activities (1 participant), sharing photos (1 participant), administrating exams orally (1 participant), searching for mathematical definitions (1 participant), showing visuals of mathematical concepts (1 participant), and developing station assignments (1 participant).

Another frequent user of the iPad in her classroom, Jasmine, described how she further planned on using the iPad as stations in her class:

> In my class, I have used the iPad in the following ways: Respond to students questions on Edmodo by using Educreations or ShowMe. Students are allowed to see the step by step advice. Students this week will get to use the tool on Monday to reflect on their learning from the unit. In one station, they will make a how to study guide on ShowMe or any app similar to it. In another station, they will make a rap using AutoRap to sum up their knowledge of the lesson. In another station, they will use notability to respond to questions for that station. Students also use Padlet to make comments about the lesson.

Even though these two quotes originated from two of the most frequent users of iPads in their classrooms, their positive views of using iPads in the classroom are representative of all participants who utilized iPads in the classroom. No participant who used iPads in their classrooms spoke negatively of their experiences with utilizing them. They only spoke of their usefulness in the classroom.

Research Question 2

Even though 15 participants used the iPad in their classroom, many encountered difficulties. Through Edmodo reflections, teachers expressed concern about their school district's acceptance of iPads. Ten participants, including 6 of the 7 participants who had not tried the iPads in their classrooms, stated there were connectivity issues between their iPads and their school's technology capabilities. The teachers did not have the right cables to connect their iPads to their document cameras, or they did not have the right technology, such as Apple TV, to make their iPads mobile in their classroom. Besides technology issues, three participants stated time was a significant concern. Tabitha felt both of these issues hindered her use of iPads in her instruction:

> I have not used my iPad in the classroom yet because of the network and because I haven't really had a moment to think about how I can best use this tool given the fact that there is only one for the class.

Another candid teacher, Tony, was reluctant to use iPads in his class, even though he expressed in his iPad assignment that he felt there was a strong need for technology in the classroom:

> To sell math, I have to understand what my customers want so I can convince them they need what I am selling. I have heard, "You should never trust a skinny cook." Why should a student listen to a teacher who will not learn new technology? This is

hypocritical to say "learn this," but I will not learn new things myself…I am in sales as a profession. I am selling a product that my buyers do not know they need, yet.

During Tony's October reflection, he stated he did try using the iPad as a tutorial but had not pushed himself yet to implement it more effectively in the classroom.

Discussion

Our findings suggested mathematics professional development with iPads created some change in many middle school mathematics teachers' beliefs about mathematics teaching. By October 2013, 15 out of 22 participants had used iPads in their classrooms with all 10 participants who used iPads in September also used iPads in October. This finding suggests that teachers who do decide to utilize iPads in the classroom may continue to do so in the future.

Though these findings are encouraging, they may not be the norm. These participants may have been more open to iPad usage in the classroom than typical middle school teachers because of their involvement in a standards-based instructionally driven professional development program, where manipulatives and iPads use are strongly encouraged. In addition to the overall supportive culture of the program's class meetings, which included iPad usage during instruction, the instructors of the course devoted time during every fall meeting to discuss how the participants used the iPads in their own classrooms, as well as responding to reflection questions based on iPads. The findings suggest that these course strategies may also have positively influenced the level to which participants actually implemented iPads with their own classes.

The findings also suggested that time and technology issues may have negatively influenced the implementation of iPads in mathematics instruction. As in prior studies, the findings from this study suggested that (a) lack of time and (b) lack of teacher support from their schools were main hindrances to teachers' use of technology in the classroom (Mumtaz, 2000). In particular, the lack of teacher support reported by participants was due to connectivity issues in the classroom as opposed to instruction with technology. Potentially, these issues could be resolved given more time. Five participants who had not utilize iPads yet in instruction did, however, state their schools were working to resolve the network connectivity problems. Of the seven teachers who did not use iPads for instruction, none mentioned negative responses from their school districts about technology usage in their classes. Even though this research is limited in scope to one particular set of teachers during a summer and part of an academic year, further research could be conducted over time to examine if the instruction changes were long-lasting.

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EVALUATING INSTRUCTION FOR DEVELOPING CONCEPTUAL UNDERSTANDING OF FRACTION DIVISION

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We designed a series of lessons geared towards promoting conceptual understanding of fraction division. We evaluated the effectiveness of these lessons by examining the nature of prospective elementary teachers' understanding of the division algorithm before and after instruction based on the measurement model of division. In this paper we share the results of our study, along with the instruments used to measure understanding of division by a fraction and the instructional strategies we used.

Many a student has parroted the phrase, "ours is not to reason why, just invert and multiply" to indicate understanding of fraction division. Procedural knowledge about fraction division is a necessary part of the mathematical knowledge for teaching, but do preservice teachers [PSTs] possess the deeper conceptual knowledge they need for their professional work in mathematics classrooms? The fraction division topic is one where many mathematical concepts find a connection from understanding fraction representation, to knowledge about multiplication and division models, and on to real-world contexts that can be used to motivate the topic. As Ma states it, "division by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic"(1999, p. 55). In discussing content topics for a mathematics course addressing number and operations, the authors agreed that fraction division was an area worth investing time and focus to see if an instructional intervention could be designed to positively impact and deepen PSTs' procedural and conceptual knowledge about fraction division. This led to the design of instructional materials, to the creation of instrumentation to gather data from PST participants, and to the analysis of the data collected. Each of the sections to follow will provide a brief literature review, details about the design and implementation of the study, the findings of the study, and a discussion of the results.

Background

The beginning point for the current study in fraction division goes back to the work of Ma (1999). While it has been almost 15 years since this research was presented, the presentation of teacher understandings on a variety of topics, including fraction division, continues to be connected to the current focus on mathematical content knowledge for teaching. Ma's results show that of 23 U.S. teachers, only 21 tried to calculate$1\frac{3}{4} \div \frac{1}{2}$, with only 9 completing the work to get the correct answer. When Ma asked the same teachers to share a good story or model for $1\frac{3}{4} \div \frac{1}{2}$, results were poor. Of the 23 teachers, 6 could not create a story and 16 shared stories that held misconceptions. She notes that "one teacher provided a conceptually correct but pedagogically problematic representation (p.64)". While this presents a bleak picture of practicing teachers, what is known about the depth of understanding of PSTs?

A study by Li and Kulm (2008) indicated that 46 middle grades PSTs at a Texas university knew the invert and multiply procedure for dividing fractions but did not know why the algorithm worked. Additionally, they were not able to evaluate whether an alternative process presented to them was mathematically correct. With evidence that improvement is needed, articles that shared information and research that would guide the creation of an instructional intervention were located and reviewed. The document, *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade* (Siegler, R, et al., 2010) provides a summary of what is known about effective interventions.

In his article, Li (2008) noted "developing students' conceptual understanding of division of fraction is not a trivial task" (p. 546). He shared an interesting comparison of the way the topic of fraction division is handled by textbooks in China versus the materials used in the United States and shared the need for a balanced perspective between the algorithmic process and the conceptual understanding. This is one of a number of well-presented articles that provide insight into strategies and instructional approaches that relate to the fraction division topic, including work by Cengiz and Rathouz (2011), Kribs-Zaleta (2008), and Gregg and Gregg (2007).

A hallmark of conceptual understanding is found in asking students to construct their own fraction division problems. Barlow and Drake (2008) shared the work of 45 sixth-grade students that were asked to respond to the following: Write a word problem that can be represented by 6 ÷ 1/2. The results from this study showed that 16% of students made no attempt to construct a problem. The authors further classified that 67% of problems written were incorrect – either for representing 6 ÷ 2, 6 × 2, or making another error. The results shared have 6% of the student-created problems as being satisfactory or extended – meaning problems were evaluated as satisfactory but also used a realistic scenario in their writing. The rubric used to score the problems submitted was based on the scoring categories from National Assessment of Educational Progress (NAEP) and refined through the examination of student work.

Research Design

Setting

This action research project was undertaken to evaluate the effectiveness of using the measurement model of division to develop both conceptual and procedural understanding of division by a fraction. The study took place at the regional university where both authors taught mathematics content courses for prospective elementary teachers. The participants in this study were all enrolled in a section of an introductory course taught by either one of the authors. All participants were informed about the study and were given the opportunity to exclude the use of their data from the study. All students enrolled in the targeted courses were expected to participate in all instructional activities as these were part of the course objectives. Instructional materials, including the pre- and posttest instruments, were first piloted during the fall of 2012. Revisions were made to the pre-and post-tests based on the pilot before being used during the primary phase of the study that took place during the spring of 2013. Sixty-six students participated in this phase of the study. The pre-test was administered approximately 5 – 6 weeks before covering fraction division and before any instruction on whole number division. The instructional intervention took place near the end of the semester during a unit on operations with fractions. Division was the last of the four operations to be explored. The post-test was administered as part of the final exam approximately 3 weeks following instruction.

Instructional Design/Intervention

The measurement model is typically used by mathematics educators as a tool for developing conceptual understanding of fraction division. For example, the text used for this course (Sowder, Sowder, & Nickerson, 2010) makes use of the idea of division as repeated subtraction (measurement division) to develop the standard invert and multiply algorithm. We based the design of our intervention on the approach undertaken in this text. Sowder, Sowder, and Nickerson (2010) began the development of the division algorithm by first considering the special case of one whole divided by a unit fraction ($1 \div \frac{1}{b}$). In doing so, one finds that there are b groups of the divisor ($\frac{1}{b}$) and then applies that unit rate to find quotients when the dividend is not 1. For example, if there are 3 groups of one-third in one, then there would be twice as many in two, giving us $2 \div \frac{1}{3} = 2 \times 3 = 6$. This argument was used to generalize division by a unit fraction as $\frac{a}{b} \div \frac{1}{c} = \frac{a}{b} \times \frac{c}{1}$. The authors then expanded this discussion to include situations in which the divisor was not a unit fraction; first considering cases in which 1 is divided by a non-unit fraction and then using the unit rate observed (number of groups of the divisor, $\frac{a}{b}$, in one whole) to solve problems in which the dividend is not one. We developed an activity within the context of pizza servings, incorporating this idea of using unit rates (how many groups of the divisor are in one whole pizza). During the instructional intervention, students used fraction circles and drew pictures to explain in writing how they determined the number of servings of each size (1/3 of a pizza, 1/6 of a pizza, and 3/8 of a pizza) they could make from first one, then two, and then additional number of whole pizzas. Following Sowder, Sowder, and Nickerson's (2010) progression, students were asked to first generalize their findings after considering problems with unit fraction divisors, before progressing to servings of size 3/8. Ultimately, the students were asked to use their findings to explain in writing how to divide any number by a non-unit fraction.

Sowder, Sowder, and Nickerson (2010) also focused on the idea of referent units in regards to both multiplication and division by a fraction. In measurement division, the dividend and divisor both refer to the same unit, whereas the quotient refers to the divisor as a whole. In contrast, under partitive division the dividend and quotient both refer to the same unit. Sowder, Sowder, and Nickerson (2010) argued that to understand a problem situation, one must know the referent unit for each quantity. Student attention to the idea of referent units occurred through activities using fraction strips and rulers to solve division problems. Students were required to explain the meaning of both the quotient and remainder. Instruction took place over two class meetings for a combined total of 2 hours and 40 minutes.

Instrumentation

The purpose of this study was to investigate the effectiveness of instructional strategies using the measurement model for developing both procedural and conceptual understanding of fraction division. We believed that mathematical knowledge for teaching fraction division involves both knowledge of the standard division algorithm as well as an understanding of why the algorithm works. Therefore, the instrument used in this study was designed by the researchers to measure both procedural and conceptual understanding of fraction division. The initial instrument had nine items involving fraction division. The first eight items were paired four division problems that were to be solved using two different methods (paper/pencil and drawing a diagram). The last item asked students to write a word problem that could be solved by dividing 4 ¾ by ½. All nine of these items were included on the pre-test during the pilot phase of the study. Since our focus was on division by a fraction, we dropped the items involving division by a whole number ($\frac{6}{7} \div 3$) as well as the items involving mixed numbers ($1\frac{3}{4} \div 3\frac{1}{2}$). The final pre/post instrument contained two division problems ($8 \div \frac{1}{4}$ and $\frac{3}{4} \div \frac{1}{2}$) that were to be solved using two different approaches (drawing a picture to solve the problem and paper/pencil). We also retained the last item asking students to write a word problem for $4\frac{3}{4} \div \frac{1}{2}$.

Data Analysis

We developed a rubric for scoring the five items on the pre- and post- test on a scale from 0 to 3. We used the same criteria to score the paper/pencil approaches taken to solve the two division problems, $8 \div \frac{1}{4}$ and $\frac{3}{4} \div \frac{1}{2}$, labeling them 1A and 2A, respectively (see Table 1). Scores on these two items were combined to obtain a measure of procedural knowledge of division by a fraction with a maximum score of 6.

Table 1: Scoring Rubric for Procedural Method 1A: ($8 \div \frac{1}{4}$) and 2A: ($\frac{3}{4} \div \frac{1}{2}$)

Score	Description
0	No attempt or un-interpretable work
1	Some mathematical basis for work shown (for example, interpreted as multiplication)
2	Minor error in arithmetic or use of decimal division
3	No errors.

A second set of criteria was applied to score the alternative approach (drawing a picture) taken by the participants in this study (see Table 2). The decision was made to give a score of 3 to student work that correctly illustrated the process of dividing by a fraction and included an appropriate explanation. Correct illustrations without explanation were awarded a 2. Illustrations which had some mathematical basis (e.g., 8 divided by 4) were awarded a score of 1.

Table 2: Scoring Rubric for Conceptual Method 1B: ($8 \div \frac{1}{4}$) and 2B: ($\frac{3}{4} \div \frac{1}{2}$)

Score	Description
0	No attempt or un-interpretable illustration
1	Some illustration/representation with mathematical meaning
2	Minor error in representation or no explanation given
3	Complete and correct illustration – explanation is given to communicate meaning

Similarly, we developed a third set of criteria to score the word problem written for $4\frac{3}{4} \div \frac{1}{2}$. A score of three was awarded for a complete and correct word problem whereas a score of 2 was given if there was an error in units or question posed, but otherwise modeled $4\frac{3}{4} \div \frac{1}{2}$. Story problems which involved an incorrect divisor or multiplication were scored as a 1. Scores on the last three items (1B, 2B, and 3) were combined to obtain a measure of conceptual understanding of division by a fraction with a maximum score of 9. A subset of size 15 was randomly selected from the pre-and post-test data (matched pairs) and scored by both researchers. We met to reconcile our scoring procedures and to refine the scoring rubric. The revised rubrics gave us a clear consensus on what score to assign to student work and documented common procedures used to solve each problem as well as the model of division represented in drawings and/or word problems written by the participants.

Findings

Frequencies were calculated for each level of performance on the five items on the pre-test and are reported in Table 3. The students were slightly more successful dividing a fraction by a fraction than they were in dividing a whole number by a fraction when a procedural approach was taken, although the reverse was true when attempting to use a diagram to find the quotient.

Table 3: Frequencies for each level of performance on pre-test items

Item	0	1	2	3
1A: $8 \div \frac{1}{4}$	17 (26.8%)	20 (30.3%)	1 (1.5%)	28 (42.4%)
2A: $\frac{3}{4} \div \frac{1}{2}$	17 (25.8%)	10 (15.2%)	4 (6.1 %)	35 (53 %)
1B: $8 \div \frac{1}{4}$	53 (80.3%)	9 (13.6%)	2 (3%)	2 (3%)
2B: $\frac{3}{4} \div \frac{1}{2}$	60 (90.9%)	6 (9.1%)	0	0
3: $4\frac{3}{4} \div \frac{1}{2}$	38 (57.6%)	25 (37.9%)	1 (1.5%)	2 (3%)

The vast majority of the participants did not even attempt to solve the problems 1B and 2B in a non-procedural manner. Twenty-eight of the participants attempted to write a story problem that could be solved by $4\frac{3}{4} \div \frac{1}{2}$, but of those, only 3 had some degree of success. The majority of the students wrote story problems that represented either 4 ¾ divided by 2 or 4 ¾ times 2 similar to what was reported by Barlow and Drake (2008). Frequencies for each level of performance on the post-test items are reported in Table 4. Level of success when using an algorithm to divide by a fraction was about the same for both items 1A and 2A. However, students were four times as successful using a diagram to divide a whole number by a fraction (1B) than when the dividend was a fraction (2B). More notable, success on this item (1B: $8 \div \frac{1}{4}$) rose from 3% on the pre-test to just over 36% on the post-test. The same swing in level of success was seen from pre- to post-test on the task of writing a word problem for $4\frac{3}{4} \div \frac{1}{2}$. However, about the same number of students wrote word problems that represented division by 2 or multiplication by ½ as was noted on work shown on the pre-test.

Table 4: Frequencies for each level of performance on post-test items

Item	0	1	2	3
1A: $8 \div \frac{1}{4}$	6 (9.1%)	7 (10.6%)	3 (4.5%)	50 (75.8%)
2A: $\frac{3}{4} \div \frac{1}{2}$	6 (9.1%)	3 (4.5%)	6 (9.1%)	51 (77.3 %)
1B: $8 \div \frac{1}{4}$	10 (15.2%)	20 (30.3%)	12 (18.2%)	24 (36.4%)
2B: $\frac{3}{4} \div \frac{1}{2}$	14 (21.2%)	41 (62.1%)	5 (7.5%)	6 (9.1%)
3: $4\frac{3}{4} \div \frac{1}{2}$	5 (7.5%)	24 (36.4%)	13 (19.7%)	24 (36.4%)

In general, students performed better on procedural items than conceptual items, both on the pre- and post-test. More students attempted all five items on the post-test than the pre-test as well. For example, the percent of students attempting to write a word problem for 4 ¾ divided by ½ rose from 28 on the pre-test (with just 2 successful) to 61 on the post-test (24 successful). The post-test was embedded in the final exam which may explain, in part, why attempt rates went up so dramatically. However, attempt rates on the procedural items were largely unchanged from pre- to post-test; suggesting students would have tried to complete the last three items if they understood the meaning of division by a fraction.

Concluding Remarks

The results reported here represent preliminary findings from our study on the effectiveness of instruction that was designed to promote procedural and conceptual knowledge about fraction division. The authors are interested in taking a deeper look at the procedures this group of preservice elementary teachers used to solve a division problem in more than one way. We have established a coding system of the variety of strategies we have observed in student work and will be taking a second look at the data collected to determine common errors and misconceptions concerning fraction division. We plan to share the results from our study with other mathematics educators with the goal of improving instruction on fraction division.

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THE ROLE OF TEACHERS' QUESTIONS IN SUPPORT OF STUDENTS' ARTICULATION OF THEIR MATHEMATICAL REASONING

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To better understand the ways in which teachers support students in articulating their mathematical reasoning, we examine the questioning practices of five high school Algebra I teachers and the students' mathematical discourse surrounding their questions. Conducted at the end of a successful six-year mathematics project and with a sample of teachers who consistently obtained high student growth on year-end measures, this study uses the framework of Franke and colleagues (2009) to analyze teacher questions and explore the ways in which different types of questions encouraged different levels of student responses.

The purpose of this study was to investigate the classroom discourse practices of Algebra I teachers from three traditionally low-performing high schools in an urban school environment. In this setting, student learning was measured according to proficiency on high-stakes tests and teacher performance was gauged through a value-added accountability model. Considering the significant changes occurring nationwide in testing practices and expectations of students and teachers, we sought to understand the ways in which teachers successful in achieving high value-added scores supported their students in learning mathematics. Since we view learning as more than proficiency on content tests, our goal was to look beyond learning as acquisition (Sfard, 1998) and to explore learning as participation in a mathematical community. To this end, we examined the classroom discourse practices in these five classrooms, focusing specifically on instances of teacher questioning in support of mathematical argumentation.

Theoretical Framework

Across the peer-reviewed literature, two prominent areas of instructional practice that relate to classroom discourse involve requiring students to justify and explain their mathematical reasoning through argumentation (Evens & Houssart, 2004; Hollebrands, Conner, & Smith, 2010; Kazemi & Stipek, 2008; Lannin, 2005; Webel, 2010; Yackel & Cobb, 1996) and encouraging students to take responsibility for their learning through questioning strategies that press students for their reasoning (Breyfogle & Herbel-Eisenmann, 2004; Cazden, 2001; Franke, Kazemi, & Battey, 2007; Franke et al., 2009; Herbel-Eisenmann, 2009; Smith & Stein, 2011; Stein, 2001). As students participate in mathematical argumentation, they work beyond simply finding an answer and move toward finding a solution, focusing on the reasoning behind that solution, and articulating their reasoning in a manner understandable to other members of the classroom community. These actions are consistent with definitions of argumentation found in the research literature (e.g., Herbel-Eisenmann, 2009; Kazemi & Stipek, 2008; Krummheuer, 1995; Lannin, 2005; Levenson, Tirosh, & Tsamir, 2009; Webel, 2010) as well as with the sociocultural view that learning occurs through engagement in practice (Lave & Wenger, 1991).

Teachers play a critical role in establishing and supporting the classroom norms that allow students to move beyond rote answers and to provide more details about their thinking. Teachers and students must both internalize their roles in participating in a community of discourse (Hufferd-Ackles, Fuson, & Sherin, 2004) and the ways in which the teacher encourages and supports students throughout this process is of vital importance. As they attempt to engage students in the discourse practices of the classroom, teachers use questions for a variety of purposes. Questions can assess student learning and also promote learning. They may elicit evidence of student thinking and can support students in making connections between topics. Teachers' use of questions to encourage students to justify and explain their mathematical reasoning is a specific focus of this study.

Characterizing Mathematics Students' Reasoning through Arguments

In his book on the uses of argument, Toulmin (1958) contends that certain basic elements compose all arguments, regardless of discipline. Though he intended his book to be "strictly philosophical," (p. vii), his model of arguments continues to gain prominence in research literature. Toulmin defines the *data* (D), as the foundation of the argument, the *claim* (C) as the conclusion to be verified, and *warrants* (W) as a bridge to explain our reasoning. For the purposes of this study, we take Krummheuer's (1995) interpretation of Toulmin's (1958) concept of argumentation. For him, argumentation is an interaction that "has to do with the intentional explication of the reasoning of a solution during its development or after it" (Krummheuer, 1995). Similar to other researchers investigating mathematical argumentation (Hollebrands et al., 2010; Singletary, Conner, & Smith, 2013; Yopp, 2013), in this study we take any verbal interaction pertaining to one or more person's reasoning for which supporting information is given as mathematical argumentation.

Characterizing Mathematics Teachers' Questions

Though the Initiate-Respond-Evaluate (IRE) pattern where the teacher *initiates* a question, a student *responds*, and the teacher *evaluates* that response (Chapin, O'Conner, & Anderson, 2009; Franke et al., 2007; Herbel-Eisenmann, 2009; Smith & Stein, 2011; Stein, 2001) has historically been the most prominent type of questioning pattern in mathematics classrooms, other patterns are emerging in contemporary mathematics classrooms. In developing their framework for analyzing teachers' questions, Franke et al. (2009) investigated the questioning practices of three elementary teachers from the same urban school. During their periods of observation, the teachers taught similar concepts and skills which allowed researchers to focus on differences in their patterns of questioning. Franke et al. expressed particular interest in the "transition from asking the initial question to pursuing student thinking" (p. 380) and worked to provide a method to look "specifically at the questions teachers [pose] in order to follow up on students' initial explanations and build on student ideas" (p. 383). Basing their coding scheme on their iterative review of the data, their final categorization consisted of five levels (see Table 1).

Table 1

Franke et al.'s Teacher Questions

Question Type	Description
1. General	Questions that are not related to anything specific that a student said
2. Specific	Questions that address something specific in a student's answer
3. Probing sequences of specific questions	Series of more than two related questions about something specific that a student said and includes multiple teacher questions and multiple student responses
4. Leading	Questions that guide students toward particular answers or explanations and provide opportunities for students to respond
5. Other	Questions that do not fit into the other categories

Note: From "Teacher Questioning to Elicit Students' Mathematical Thinking in Elementary School Classrooms," by M. Franke et al., 2009, *Journal of Teacher Education, 80*, 380-392.

For this study, we utilized Krummheuer's interpretation of Toulmin's (1958) model of argumentation along with the questioning framework developed by Franke et al. (2009) to guide our inquiry with two research questions:

1. Based on Krummheuer's interpretation of Toulmin's (1958) model of argumentation, what is the nature of mathematical argumentation in these classrooms?

2. Based on Franke et al.'s (2009) framework, in what ways do teachers' questions support students' articulation of their mathematical reasoning through argumentation?

Methods

To examine the nature of mathematical argumentation emerging in mathematics classrooms and the role of teachers' questions in support of this mathematical reasoning, this study followed a case study research design (Merriam, 1998; Miles & Huberman, 1994; Stake, 1995). A case study research design may be composed of one or several cases (Merriam, 1998; Miles & Huberman, 1994; Stake, 1995) and multiple-case studies, defined as those studies incorporating two or more cases, often yield a more compelling and robust interpretation of findings. Thus, this study used a multiple-case study design to maximize the likelihood of observing and understanding mathematical argumentation and the methods by which it is fostered in these mathematics classrooms (Stake, 1995). It explored in depth both the nature of and the teachers' supports for episodes of mathematical argumentation in an attempt to gain greater understanding of their purpose and meaning to those involved (Merriam, 1998).

Sample

The five teachers selected for this study were a part of a larger six-year longitudinal project initiated to recruit and retain qualified mathematics teachers in ten traditionally low performing high schools in a school district in a southeastern state. The school district is one of the largest school systems in the state and one of the fifty largest districts in the country. With a diverse student population spread out over urban, suburban, and rural areas, the school system has more than 70,000 students. Sixty percent of the student population in the district is non-white, and approximately 55% of the students receive free or reduced-price lunches. Eight of the ten project schools are classified as urban, with the remaining two considered suburban. By stabilizing the rate of teacher attrition and supplementing teachers' knowledge both in terms of mathematics content and pedagogy, the project intended to increase student mathematical learning and thus raise student test scores. The five participant teachers were a part of the larger project since its beginning, and all five had consistently received the highest value-added scores for their students' performance on statewide standardized tests.

Data Collection

Following qualitative research tradition, we conducted classroom observations with each of the five participant teachers, observing the same Algebra I class for five consecutive ninetyminute sessions of instruction, totaling 7.5 hours of observations per participant. Following the week of observations, we then interviewed each teacher to discuss the observations in order to understand the reasons behind their instructional choices.

Data Analysis

The data analysis for this study consisted of both within-case and cross-case analyses (Merriam, 1998; Miles & Huberman, 1994). Beginning with a within-case analysis of each teacher's observations, we completed detailed descriptions and an analysis of the mathematical argumentation occurring in each classroom and the teacher questions prompting each occurrence. An initial review of the field notes and observation videos identified 53 episodes of talk where discussions about mathematics occurred between teacher and students around specific problems or topics. Two additional iterative passes through those episodes yielded 19 episodes of talk we classified as mathematical arguments, the probing sequences of questions (Franke et al., 2009) surrounding those arguments, and detailed transcriptions and models for each episode. We then conducted a cross-case analysis in an attempt to understand patterns that transcended the five cases. Cross-case analysis helped to strengthen our understanding of the commonalities and differences between the mathematical argumentation occurring in the five classrooms and the purposes for which teachers fostered such discussions and supported them through their use of questions.

Results

Our findings indicate that, while multiple questioning types are used, questions that help extend student's mathematical thinking are rare. Our analysis of mathematical argumentation reveals that most arguments occurring in these classrooms contain only the basic components of data, claim, and warrant (Toulmin, 1958). By requiring the presence of both claim and data to designate a discussion an argument, these episodes provide insight into both the students' articulation of their mathematical reasoning and the teachers' questions surrounding the students' statements. To prompt students to supply data in support of their claims, the teachers used the simple question, "Why?" in 10 of the 19 arguments. In 5 of the 19 arguments, the teachers asked "How do you know?" to encourage students to continue beyond the statement of the claim. In Franke et al.'s (2009) categorization, these questions each potentially represent the first in probing sequence of specific questions. Table 2 shows the breakdown for each teacher.

Table 2

Descriptions of Teachers' Prompts for Data

Teacher	Why?	How do you know?	Other questions
Abby (2)	0	0	2
Denae (3)	0	2	1
Kendra (6)	3	2	1
Leslie (3)	2	1	0
Will (5)	5	0	0
TOTALS (19)	**10**	**5**	**4**

In 12 of the 19 episodes, following the students' statement of data, the reasoning connecting the claim and the data is left unstated. These warrants, as Toulmin (1958) defined them, were left to be inferred by the students in the class. In the remaining seven episodes, students explicitly stated the warrants between their claims and data. In five of those episodes, the teachers pressed the students for the connection between the claim and data by asking either "Why?" or "How do you know?", using the same types of prompts as those with which they elicited data.

In the observed episodes of argumentation, the teachers recognized a need for students to move beyond simply offering answers to questions with no elaboration about their reasoning and acknowledged the importance of prompting for additional information. While the arguments that contain explicit warrants provide indications of the teachers' developing abilities to engage students in mathematical discussions where the students themselves elaborate on the data justifying their claims in the form of explicit warrants, we conjecture that conducting probing sequences of questions to elicit students' thinking remains challenging for the teachers.

Conclusion

Our findings suggest a need for greater support from the mathematics education community for prospective and practicing teachers in learning instructional practices that support classroom discourse that engenders students' abilities to justify and explain their reasoning. Future research should include an examination of teachers' intentions and beliefs regarding students' abilities to participate in mathematical argumentation and the learning supported by such practices.

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LOOKING FOR ELEMENTARY MATHEMATICS TEACHERS' COMMON CORE-FOCUSED INSTRUCTION

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This manuscript describes six elementary teachers' instructional changes through the lens of the Standards for Mathematical Practices (SMPs). Teachers were randomly selected from a larger sample of K-5 teachers who engaged in yearlong professional development targeting the SMPs. Videos of their pre- and post-professional development programs were examined using a SMPs-focused protocol. They overwhelmingly provided more opportunities for students to engage in the SMPs after the professional development experience. We connect this impression with ways to effectively foster elementary teachers' SMP-focused instructional practices through professional development.

Related Literature

Standards for Mathematical Practice

The Common Core State Standards for mathematics (CCSSM; Council of Chief State School Officers [CCSSO], 2010) will require teachers to reevaluate their current instruction (National Council of Teachers of Mathematics [NCTM], 2010). There are two halves to the standards: Standards for Mathematics Content and Standards for Mathematical Practice (SMPs). The SMPs offer characterizations of behaviors and habits that students should demonstrate while learning mathematics. The *Principles and Standards for School Mathematics* (NCTM, 2000) and *Adding it Up* (Kilpatrick, Swafford, & Findell, 2001) guided the descriptions of the SMPs. NCTM's (2000) process standards are problem solving, reasoning and proof, communication, connections, and representation. The notion of mathematical proficiency includes conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive disposition (Kilpatrick et al, 2001) in order to descriptively derive the notion of what is necessary for mathematical proficiency. Unfortunately, the promotion of these proficiencies is not evident in every classroom. Thus, professional development must be designed to enhance teachers' understanding of the SMPs and support them to enact mathematics instruction focused on them. These behaviors are not isolated and often occur in tandem with one another because they are interrelated behaviors (CCSSO, 2010). For example, modeling with mathematics and attending to precision are likely to occur during a modeling-focused activity. Students are expected to reflect on their mathematical models and revise them as needed, which likely occurred because they saw a way to more precisely (e.g., effectively or efficiently) describe the mathematical situation embedded within the task. In order for students to engage in the SMPs like this, teachers must design and enact instruction that allow students to wrestle with mathematics content and its applications in an environment that supports and sustains meaningful engagement with mathematics.

The literature is clear about teachers' instructional emphasis of the process standards or mathematical proficiency: it is not occurring often (Hiebert et al., 2005). Hence, mathematics teacher educators have tried to support instructional growth by providing long-term (i.e., one year or more) of professional development that assists K-12 mathematics teachers' understandings of mathematical proficiency and/or the process standards (e.g., Anderson & Hoffmeister, 2007; Boston, 2012; Boston & Smith, 2009). These recent studies and others describe ways to benefit teachers' instruction but there is a noticeable gap when the literature focuses on instruction in the CCSSM era. The purpose of this paper is to build upon the current literature base as a means to discuss K-5 mathematics teachers' instruction, specifically focusing on the ways they provide students' opportunities to engage in the SMPs.

Professional development: What works

A metaanalyis of PD suggests that there are some key features to designing effective inservice teacher education (Guskey & Yoon, 2009). Two of those five features include (a) professional development (PD) activities that encourage teachers to adapt a variety of practices to a content area rather than encouraging a set of best practices and (b) PD activities that encourage teachers to try ideas in their classroom. Boston (2012) details how focusing on implementing worthwhile tasks during a yearlong PD enhanced secondary teachers' knowledge, which in turn influenced their instructional practices. For example, after the yearlong PD they were able to identify elements of tasks with high cognitive demand and concurrently selected more tasks with high cognitive demand for their own instruction. Improving teachers' ability to select worthwhile tasks is not the only way to impact their instructional outcomes (Boston & Smith, 2009); supporting them to establish an effective learning environment and sustain mathematical discourse between students are also necessary to maximize students' opportunities to learn (NCTM, 2007). Building upon this foundation for effective PD, a yearlong project was conducted in a Midwestern state to prepare teachers to implement the CCSSM. We aim to explore how teachers' instruction changed to support students' engagement in the SMP and attempt to connect their growth to the PD project. Our research question was: How does teachers' mathematics instruction evolve during the PD? Further, we wondered how teachers' changes might be related to three central areas of this PD: learning environment, worthwhile task, and discourse. We examined K-5 teachers pre- and post-PD mathematics teaching specifically looking for specific instructional actions that are connected to the SMP.

Method

Context of the Professional Development

We focus on K-5 teachers' experiences as influenced by a yearlong grant-funded professional development program. Teachers met four times for four-and-a-half hour sessions between March – April 2012. We met for eight 8-hour days during the summer and then met twice face-to-face for four-and-a-half hour sessions between August – October 2012. Teachers were provided with numerous online assignments that were intended to facilitate further online interactions between March – October that might support teachers' understanding of the SMPs. Generally speaking, the aim of the PD projects included (1) making sense of the SMPs, (2) exploring inquiry through three broad areas consisting of worthwhile tasks, mathematical discourse, and appropriate learning environments, (3) implementing classroom-based tasks that aligned with the CCSSM, and (4) increasing mathematical knowledge and understanding. Teachers read and reflected on their instruction as well others implementing CCSSM-aligned mathematics instruction. Teachers read and discussed chapters from NCTM books (e.g., Mathematics Teaching Today [2007]) and completed various assignments including reflective journaling, writing, enacting, and reflecting on CCSSM-aligned mathematics lessons, and solving mathematics problems.

Participants

This project served 23 grades K-5 mathematics teachers and at least three teachers representing each grade level. Teachers came from urban, suburban, and rural school districts. We decided to randomly sample one teacher from each grade level for this initial study. We had no reason to believe that one teacher grew more than another during the yearlong professional development and random selection provided us with greater opportunities to characterize that growth compared to purposeful selection. Table 1 provides the years of teaching experience and school district context for each randomly selected teacher. We intend to explore all teachers' instruction after exploring this random sample of teachers as our pilot project.

Table 1. Participants' demographic information

Grade-level	Gender	Years Experience	District Context
Kindergarten	Female	13	Urban
First Grade	Female	12	Rural
Second Grade	Female	14	Rural
Third Grade	Female	12	Urban
Fourth Grade	Male	13	Rural
Fifth Grade	Female	6	Rural

Procedures

Data Collection and Analysis

Teachers were asked to design, enact, and videotape one lesson during the Spring 2012 and Fall 2012 semesters. The Human Subject Review Board indicated that only teachers' consent was required since the focus of our study was on teachers' instruction. Teachers consented to videotaping one lesson and sharing the video with us for analysis. Depending on the grade level and the local school context of the teacher, the videos were as short as 25 minutes and as long as 65 minutes. Since our study focused on ways that teachers supported students' engagement in the SMPs during instruction, we investigated the videotapes as a means to best report any instructional changes made during the PD program. Such analysis approaches have been used in similar studies such as Boston (2012) and Boston and Smith (2009).

Data analysis required two parts. The first part was composed of two stages. The first stage was watching the videotapes and reflecting on instruction using a protocol focused on the ways that teachers' instruction supported engagement in the SMPs. Two mathematics education faculty watched the videotapes and conducted the analysis. The protocol used for analysis was developed by Fennell, Kobett, and Wray (2013). It provides look-fors that link mathematics instruction with behaviors and actions that are associated with the SMPs. For example, three aspects were used for the first SMP: Make sense of problems and persevere in solving them. They included (a) Involve students in rich problem-based tasks that encourage them to persevere in order to reach a solution, (b) Provide opportunities for students to solve problems that must have multiple solutions, and (c) Encourage students to represent their thinking while problem solving (Fennell et al., 2013). While there may be other aspects indicative of SMPs, the protocol provides an evidence-based framework for examining mathematics instruction using the SMP lens. The second stage occurred after watching a video. We compared our coding observations with one another, pausing as needed to discuss areas where we differed. If needed, we watched the video a second time. Discussions ended when both coders agreed that there was sufficient evidence related to a look-for. The second part of data analysis focused on making sense of the data to answer our research question. We intended to quantify changes in the number and type of instructional opportunities related to the SMPs. This was accomplished by examining our evidence in two ways. The type and frequency of instructional opportunities related to each SMP were categorized. Then we explored the changes in instructional opportunities related to the SMPs across teachers with the goal of generating general impressions. After considering the data, we drew out general impressions that are shared in this manuscript.

Results

Overall, teachers provided more instructional opportunities intended to engage students in the SMPs. Figure 1 shows the frequency of instructional opportunities for each SMP during the pre- and post-PD instructional lesson.

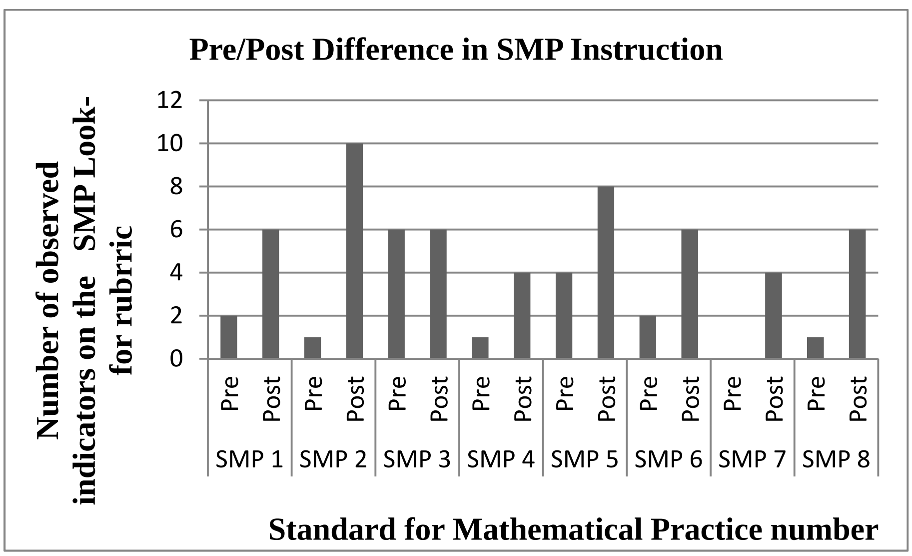

Figure 1. Frequency of observed indicators in Pre- and post-PD instruction

The participants showed an increased promotion of every SMP with the exception of SMP 3, which remained constant. The median and range for the frequency of codes was 1.5 opportunities and [0,6] for the pre-PD instruction and was 6 opportunities and [4,10] for the post-PD. These quantitative findings suggest that on average, teachers provided more opportunities for students to engage in the SMPs after the PD. Looking specifically at each teacher revealed that every teacher provided more opportunities to engage in the SMPs. We sought to qualitatively understand these changes with respect to the SMPs and three PD factors: learning environment, mathematical task, and discourse. Due to the brevity of this proceedings manuscript, we are only able to provide qualitative description of one teacher's instructional changes.

We noticed that instructional opportunities were clearly influenced by the implementation of their choice of task, changes in learning environment, and ways discourse was promoted. For example, the second-grade teacher's pre-PD instruction focused on guiding students through the definitions of a fraction in the context of exercise-laden teaching. Students were seated in rows and asked to follow her model of using pattern blocks to represent benchmark fractions. Then, students watched a video stemming from her textbook showing exactly the same activity as her students completed just minutes ago. Finally, students worked on a series of exercises without using pattern blocks. Students spoke only when the teacher asked a question. This directed instruction approach stands in stark contrast to her post-PD instruction.

The post-PD warm-up task was to determine how many letters there were in sum of the first names of the class. Students were seated in small groups and had access to a variety of manipulatives on their desks. The teacher encouraged several students to share how they counted the letters. After the warm-up task, she asked them to determine the number of legs in the classroom. The teacher utilized a think-pair-share approach with this task. Students used an initial representation (e.g., symbolic, graphical, verbal, and/or concrete) to solve this task and the teacher monitored students' work. She reminded students to explain what they were doing on their papers and to be prepared to justify why their approach is effective and efficient. As students finished working with an initial representation, she asked them to employ another viable representation to solve the problem. Finally, students shared how they solved the problem using multiple representations and then justified their strategy to a partner and then the class. Students also responded to questions from the teacher but the flow of discourse included multiple student-to-student interactions as well. It was apparent how the teacher provided an opportunity for her students to decontextualize the mathematical elements from the task and later contextualize the mathematical symbols with the referents in the problem. Through these instructional changes and ones like it, our sample of teachers provided greater instructional opportunities for students to engage in the mathematical practices.

Implications

From this study, we learned that teachers overwhelmingly engaged in greater opportunities related to the SMP after the PD then before it. These changes are associated with modifications to the learning environment, mathematical task, and/or ways that the teacher initiated and sustained mathematical discourse. For example, the second-grade teacher's post-PD changes are tied to all three instructional aspects. These changes led to greater opportunities to foster students' engagement in the SMPs. We cannot link one aspect of the PD with the changes but are able to suggest that yearlong PD focusing on the Common Core and our three central instructional aspects did lead to changes in the way these K-5 teachers designed and implemented mathematics instruction. The SMPs do not dictate curriculum or teaching but they do provide ideas for mathematically engaging students in classroom instruction. PD may help mathematics teachers at all grade levels make sense of mathematics instruction that supports students' appropriate mathematical behaviors.

This study has a second implication. Results from it support the prior literature suggesting that yearlong PD, which adheres to what works for designing and implementing effective PD, tends to lead to instructional changes that promote improved opportunities to learn.

Limitations

Qualitative approaches allow researchers to draw on their lenses and frames of reference to make sense of experiences in the world. The results offered here are not generalizable to all teachers and are particular to this set of teachers. Our sample also limits some of the findings. That is, teachers volunteered to participate in the PD and those who are less motivated to complete yearlong PD may have different outcomes making instructional changes. Furthermore, teachers differing in some way from our greater K-5 sample in terms of years of experience, school district location, or other aspects might lead to other findings. A third limitation was that the pre-PD video was done after nine hours of Common Core PD. Thus, any growth in teachers' promotion of the SMPs is limited because they experienced some PD prior to their pre-PD instructional video.

Conclusion

The third limitation provides an important finding about the importance of our yearlong Common Core PD program. Teachers had another 78 hours of PD following their pre-PD videos, which is a strong indication of the impact sustained PD has on teachers' instructional outcomes. That is, teachers provided limited opportunities for students to engage in the SMPs after nine hours of PD, yet improved greatly after more time to consider their PD experiences and translate them into pedagogical instantiations to promote the SMPs. The evidence found in this study suggests that K-5 teachers benefitted from reflecting and working to implement the CCSSM through three instructional areas: learning environment, mathematical task, and mathematical discourse.

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THE PROFESSIONAL NOTEBOOK AS A VEHICLE FOR CONTINUED GROWTH

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The ultimate goal of professional development (PD) is to positively impact student learning. One way to achieve this is to help teachers develop habits of mind which include the willingness to question, experiment, and reflect (Doerr, Goldsmith, & Lewis, 2010). This qualitative, multi-case study investigated how four teachers used a professional notebook (PN) as a vehicle for growth. The PN is a place to set goals, collect evidence of best practices, evaluate student learning, and plan for continuous improvement. Findings indicated teachers used their PN to organize their PD materials, reflect on their teaching, and analyze student work.

The main goal of all professional development (PD) is ultimately to improve student learning. Typical PD requires teachers to be passive recipients of information. Teachers often do not have dedicated time to reflect on newly acquired knowledge and how to integrate it into their current practices. Using a portfolio could be one way to integrate PD with practice. While portfolios can be used by administrators to assess teachers, they can also be used as a powerful growth tool. Portfolios, or professional notebooks (PNs), can be used to develop critical thinking, allow for continuous reflection, and lead to well informed decision making.

Literature Review

Professional Development

Professional development is a requirement of every K-12 public school teacher in the United States. PD may involve one-day trainings, conferences, university courses, online learning, or site-based activities such as study groups, action research, and coaching. Sztajn, Marrongelle, and Smith, in their report on recommendations for PD (2012), noted “A current problem with PD is that available opportunities are frequently fragmented and episodic… in part because PD is supported and coordinated through many different types of organizations” (p. 14). PD often comes as a mandate based on school, district, or even statewide initiatives. Pianta (2011) stated, “It is a travesty that despite districts spending thousands of dollars per teacher…, these dollars are most often spent on models that are *known to be ineffective*. These are predominantly one-time workshops… or models that have little basis in what is known about effective instruction, curriculum, or classroom interactions” (p. 4).

Despite these challenges, the literature does provide evidence of models that can be effective. The 2010 NCTM Research Brief includes the following goals of PD for mathematics educators: build teachers' capacity to notice, analyze, and respond to student thinking; build productive habits of mind; and build collegial relationships, and support continued learning (Doerr, Goldsmith, & Lewis, 2010). Doerr et al. (2010) recognized that these goals are best met when PD occurs over a period of time, involves systemic support, and engages teachers in active learning.

Knowledge is most likely to be retained if it is situated (Putnam & Borko, 2000). "Knowledge about teaching and learning only makes sense when considered in the context of a teacher's own school culture and climate" (Maggioli, 2004, p. 11). Maggioli echoed the recommendations of Doerr et al. (2010) that teachers need to be active participants in their growth. Effective PD supports educators' increasing awareness of themselves, their students, and the environment in which they work.

Metacognition

In *Principles and Standards* (2000) NCTM underscored the importance of metacognition:

> Effective problem solvers constantly monitor and adjust what they are doing. They periodically take stock of their progress... If they decide they are not making progress, they stop to consider alternatives and do not hesitate to take a completely different approach. (p. 53)

Students often have the misconception that mathematics is about learning a set of procedures, formulas, and methods. Research has shown that even college students will work at a problem for an average of 2.2 minutes, and then give up if they do not arrive at a solution (Schoenfeld, 1992). Effective teachers model true mathematical thinking which involves a dynamic interplay among the following processes: reading, analyzing, exploring, planning, implementing, and verifying a solution to a problem. The PN can be a tool for developing metacognition.

Professional Notebook

While portfolios are commonly used in teacher preparation programs (Zeichner, & Wray, 2001), inservice teachers rarely continue this practice voluntarily. Given the current political climate of teacher accountability, if portfolios are used, they are often evaluative in nature. The PN (referred to as portfolio in the literature) can provide a medium for reflecting on one's practice; reflection leads to critical thinking and more informed decision making. Through the use of a PN, teachers can identify personal strengths and weaknesses and look at gaps between PD and practice (Khan & Begum, 2012). The PN also allows teachers to express their creativity and diversity. It is a powerful organizational tool that enables an educator to track growth over time. It also can be a way of exchanging ideas as teachers share with peers. In light of the literature reviewed, two research questions guided this study: How can the PN be used as a vehicle for growth as a mathematics educator? How does the PN allow teachers to notice and analyze student thinking, as well as develop habits of mind (defined in part by a willingness to experiment and question) that lead to continued improvement?

Methodology

The design for this research was a descriptive multi-case study. The case study method was chosen because the research involved a real life context where the researchers had no control over the events (Yin, 2003). Each case is defined as the teachers' PN.

Setting

The study took place at a Hispanic-Serving Institution in the southern U.S. The setting was a two-year (2011-2013) grant funded PD program for Algebra teachers. During the summers, teachers (n=38) participated in 40 hours of graduate class work as well as 40 hours of independent work on campus. This time was dedicated to developing teacher-led reform projects. The grant also included four PD days each academic year, as well as four observations from a designated mathematics instructional coach. Additionally, teachers had the opportunity to create a PN each year. The intent of the PN was formative and not evaluative in nature.

Participants

In the second year, 26 teachers and two district coaches – collectively representing 14 middle or high school campuses – participated in the grant. The first year of the grant, teachers were told they would be compensated $1,000 if they completed the PN. The second year of the grant, compensation was based on the number of teachers who completed notebooks. Of these 28 participants, nine completed PNs, and seven of these nine teachers completed PNs both years. From these seven participants, four were purposely selected for this study to represent both middle and high school levels. Additionally, these teachers were from different types of campuses: International Baccalaureate (IB), early college high school, and traditional.

Data Collection

Data consists of PNs created by the participants in the second year and answers to open-ended questions. While the literature refers to portfolios, we use the term professional notebooks to emphasize the reflective aspect of this tool. The guidelines for creating the PNs included these components: table of contents, (at least three) personal goals for the year, sample student work with teacher reflection, alphabetical glossary/index by topic, and an optional vocabulary list (professional and/or content specific). In Summer 2013, participants were asked three open-ended questions regarding the usefulness of the PNs: 1) Did keeping a professional notebook help you to grow as an educator? 2) If so, how did it help you to grow (you can include things you learned about yourself, your students, teaching)? 3) Are there things you learned last year through keeping a professional notebook you are implementing in your classrooms this year?

Data Analysis

We analyzed the data using the four goals outlined in Doerr et al. (2010) as a framework, because these PNs were products of a grant-funded PD program, beginning with summaries of findings from each notebook. We did independent analysis then used researcher analysis triangulation to verify accuracy of information.

Findings

Four Case Studies

Maria. Maria is a teacher with 15 years of experience. She works at an early college high school that is on block scheduling. This schedule allows extra time for planning and PD, including Friday afternoons. During the 2012-2013 academic year, Maria's school had a campus-wide goal of incorporating writing across the curriculum. Maria used the PN as a tool for collecting and organizing the materials from the campus PD as well as the grant PD. One technique she used was a "Dear Confused" letter, where students would write a letter to a confused hypothetical peer explaining a mathematical process. She reflected, "I really like this activity… next year I will pair up a Dear Confused with a Dear Expert."

Within her PN Maria used feedback from observations to inform her instruction. For example, her coach pointed out that students in an Algebra II class were struggling with simplifying rational expressions because they did not have a strong foundation in finding common denominators. Maria reflected within her notebook, "very true...cannot assume the student knows or remembers so implementing more reviews as needed." When asked how Maria used her professional notebook, she replied,

> If used properly it provided reflection time and so as an educator – I could reflect on how I would change things to help students' learning process; what I would have done differently; and what I would want to do in the future. Also it can be used as a reference now and in the future.

Elena. Elena has 15 years of experience teaching and currently works at an IB school. She was the only teacher to create a Professional Interactive Notebook (PIN). The PIN is designed such that the right side contains input; these are notes taken by the student or hand-outs given by the instructor. On the left side of the PIN, students are able to graph, draw, respond, and reflect. This provides multiple ways of processing the information they have received (Carter, Hernandez, & Richison, 2009). The PIN incorporates both left and right brain activities, allowing for active engagement in learning.

In her PIN, Elena included goals related to writing, engagement, and student achievement on the yearly standardized test. On the right side, the teacher included results of formative assessments, handouts from PD activities, student work samples, and results of students' reflections. On the left side, there are samples of Foldables (Zikes, 2003) created by the teacher and reflections on classroom observations, as well as responses to student work.

She incorporated hands-on activities. One example is the Tin Solid Project. This small group activity involves the students creating a structure out of nine geometric solids. The students are required to calculate how much tin foil they will need to completely cover their structure. This project is a problem-solving activity with multiple solutions. Elena utilized rubrics to grade the students and then had the students complete a self-assessment.

Elena shared about her PIN, "I have already made changes to some of my lessons based on my reflections about my students' work. It also gives me a sense of pride that I've created this masterpiece all by myself. I want to use it."

Susan. Susan is a veteran educator with 26 years of experience teaching at both the secondary and college levels. She teaches Algebra and Geometry to eighth graders in a magnet gifted IB school. Her goals for the year included using a variety of resources and groupings (individuals, pairs, small groups) with a focus on problem-solving and the use of application level activities within her class.

Her PN showed detailed focus on student thinking. She had entries on 21 separate activities, with reflections on what worked, difficulties the students encountered, and ways to improve. Her notebook included both global evaluations of a class' understanding of a topic, as well as analysis of specific student errors as shown in Figure 1 below.

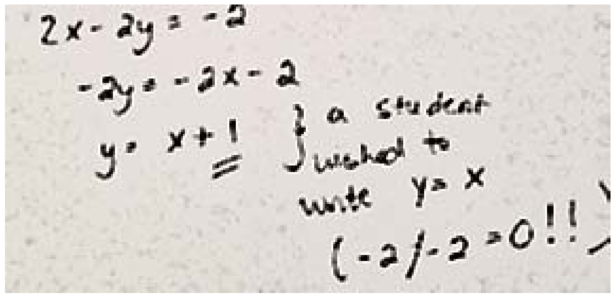

Figure 1. Teacher noticing student error.

Susan reflected, "The notebook helped me analyze student understanding. I am becoming more comfortable with student struggle… I want the lesson to move along, but I try to keep the thought of the necessity of student struggle in mind."

Her PN included students' work on several problem-solving activities, including one called Sailor Monkeys (Bair, & Mooney, 2013). Three of the student solutions used a variety of strategies: drawing pictures, working backwards, and finding patterns. Susan found that the practice of keeping a notebook motivated her to look for more engaging lessons. "As the year progressed, I began to think of lessons in terms of how I would reflect on them and what kind of lessons would best be showcased in a notebook." Even after the grant ended, this teacher continued a relationship with the coach, inviting her back to co-teach a lesson.

Allison. Allison is a teacher with nine years of teaching experience. She currently teaches at an inner city high school, where just getting students to regularly attend school is a challenge. Allison focused on student motivation. Her PN included the teacher-developed survey seen in Figure 2 below. This survey consisted of 15 Likert scale questions and one open-ended question, "What can I do to help you be more successful in math?" Allison's use of this survey illustrates that she values student feedback.

Allison documented high energy games and group activities. She used a Wheel of Fortune game and Trashketball, where students worked cooperatively to solve problems before shooting baskets with paper or spinning a virtual wheel. Her notebook also had evidence of several projects, including one about linear functions from her Algebra class and a kite project from her Geometry class.

Allison was highly receptive in the coaching process. She made detailed reflections before and after each observation. Allison reported, "I learned that I do implement a lot of creative things in my classroom… I believe the critiques… helped keep me motivated and gave me inspiration to keep looking for and trying new ideas in my classroom."

Scale:

1=Strongly disagree **3=agree** **5=Strongly agree**

1. My teacher makes an effort to get to know me.
2. I feel that my teacher recognizes my effort and progress
3. I get to participate in different groups with different classmates in ways that help me learn.
4. I get different opportunities to show my understanding, learning and skills (besides tests).
5. My teacher gives me a chance to revise or correct my work.
6. If I don't understand something, my teacher provides me with the time and chance to re-learn the material.
7. In my class, I clearly know what I'm learning and why I'm learning it.

Figure 2. Teacher-created student survey.

Discussion

These four case studies provide evidence that the PN could be used as a powerful tool for professional growth. The teachers used the PN as an organizational tool for PD materials, as a means to reflect on student learning, and as a place to set goals and document progress on those goals. Portfolios have often been used as an alternative form of assessment or as a way for teachers to obtain employment, with a focus on an external audience. This study demonstrates that there is value in creating a portfolio purely as a tool for personal growth. Teachers can be honest in their self-assessment if they know the portfolio will not be used for evaluative purposes, learning from both their successes as well as their failures.

The teachers identified organization as one of the most useful aspects of keeping a PN. As teachers reflect on their teaching, they can save effective lessons and sample student work in the PN for future reference. The accountability of the grant, and monetary compensation, were extrinsic motivators. In the absence of funding, one suggestion is teachers could form communities of practice to hold one another accountable. Teachers could share ways in which they were utilizing the PNs, as well as insights gained through this practice.

It is worth noting that three of these four teachers were at a school where they had two planning periods daily. One implication from this study is the importance of systemic support for growth – release time for PD and adequate protected daily planning time. When given this support, teachers can build professional notebooks that allow them to reflect and improve their practice, which ultimately could lead to improved student learning.

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A CONCEPTUAL MODEL FOR ALGEBRA TEACHER SELF-EFFICACY

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There is empirical evidence that algebra is a major predictor of academic success in college and career readiness (Adelman, 2006; Stein, Kaufman, Sherman, & Hillen 2011). Producing algebra teachers who are confident and competent in their abilities is essential for student success. A group of mathematics educators banded together to address a gap in the research on teachers' sense of self-efficacy in knowing and teaching algebra. This paper focuses on a conceptual model for algebra teacher self-efficacy that forms the basis for developing a survey instrument to measure teachers' self-efficacy in knowing and teaching algebra.

There is empirical evidence that algebra is a major predictor of academic success in college and career readiness (Adelman, 2006; National Governors Association (NGA) & Council of Chief State School Officers (CCSSO), 2010; National Mathematics Advisory Panel, 2008; Stein, Kaufman, Sherman, & Hillen 2011). Producing algebra teachers who are confident and competent in their abilities to teach algebra is essential. A team of mathematics educators from nine Texas universities/colleges (Baylor University, University of North Texas, Texas State University, Sam Houston State University, Texas Woman's University, Texas Christian University, Texas Wesleyan University, LeTourneau University, and Tarrant County College) collaborated to develop a conceptual model for examining algebra teacher efficacy. This paper describes the foundation for development of that conceptual model.

Theoretical Framework

Teacher Knowledge

Historically, teacher knowledge in mathematics, as measured by teacher certification, courses completed, major, grade point average, or scores on standardized exams, did not have a significant relationship to student attainment, especially at the elementary level (Grossman, Wilson, & Shulman, 1989; National Mathematics Advisory Panel, 2008). These results imply that traditional mathematical content knowledge, although necessary, is not sufficient for tasks involved in teaching mathematics (Hill & Ball, 2009). Shulman (1986) proposed that, in addition to subject matter knowledge, teachers must also have pedagogical content knowledge (PCK).

Shulman (1986, 1987) recognized that his framework for teacher knowledge needed further development, especially in the realm of content knowledge. In the field of mathematics education, Ball, Thames, and Phelps (2008) built upon Shulman's work through extensive analyses of classroom practice. They identified tasks for teaching mathematics (e.g., modifying tasks for differentiated instruction, evaluating student explanations) and the associated mathematical knowledge needed for carrying out these tasks. They further refined the various components of their content knowledge framework, namely Mathematical Knowledge for Teaching (MKT), through the creation and use of measures of MKT. MKT, which has two major domains (Subject Matter Knowledge and Pedagogical Content Knowledge), closely aligns with Shulman's (1986, 1987) categorization of teacher knowledge (Table 1). The more specific classification of content knowledge in Table 1, common content knowledge and specialized content knowledge, is a major contribution to the field of teacher knowledge in mathematics education (Ball et al. 2008). In simple terms, common content knowledge is mathematical knowledge that is used in areas other than the field of teaching whereas specialized content knowledge is unique to teaching mathematics. Mathematics teachers must know more than the content of the school curriculum to effectively conduct the previously mentioned tasks of teaching. Studies involving MKT have found a significant relationship between teachers' MKT and their instruction quality (Hill et al., 2008) and the achievement of their students (Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2008).

Table 1

A Comparison of MKT and Shulman's Categorization of Teacher Knowledge

Mathematical Knowledge for Teaching	Shulman's Categorization of Teacher Knowledge
Subject Matter Knowledge	
Common Content Knowledge	Content Knowledge
Specialized Content Knowledge	Content Knowledge
Horizontal Content Knowledge	Curricular Knowledge – Vertical Knowledge
Pedagogical Content Knowledge	
Knowledge of Content and Teaching	PCK - Representations
Knowledge of Content and Students	PCK – Student Conceptions/Misconceptions
Knowledge of Content and Curriculum	Curricular Knowledge – Knowledge of Curricular Programs and Instructional Materials

Teacher Self-Efficacy

Research on pedagogical content knowledge has lacked focus including the relationship to teacher beliefs (Ball, Thames, & Phelps, 2008). Enochs, Smith, and Huinker (2000) also found an inconsistency in the research on teacher preparation dealing with beliefs of pre-service teachers, and emphasized that both content knowledge and teacher beliefs are important and should be a part of teacher preparation programs. Therefore while teachers do need deep mathematical content knowledge and particular understandings in mathematical pedagogical knowledge, they also need a strong belief in what they can do as teachers of mathematics, having confidence in their ability to reach all students and assist them in being successful in mathematics; that is, a sense of self-efficacy.

However, several research studies in the area of self-efficacy have not produced the desired results with respect to its effects because of the lack of specificity of the self-efficacy measure (Pajares, 1996; Usher & Pajares, 2008). Self-efficacy beliefs are not always consistent among the various teacher responsibilities and subject areas (Bandura, 1997). Usher and Pajares (2008) contend that self-efficacy measures function best when targeted at appropriate levels of specificity. This line of reasoning could be extended to imply that self-efficacy beliefs towards mathematics in general may be different than those concerning a specific aspect of mathematics such as algebra. Additionally, according to Finney and Schraw (2003), self-efficacy is task-specific (e.g., teaching students how to simplify an algebraic expression) rather than domain-specific (e.g. teaching algebra). They argue that even though the domain-general self-efficacy is somewhat generalizable to specific tasks within that domain, “the closer the correspondence between the task and self-efficacy assessment, the better the prediction of performance on the task” (Finney & Schraw, 2003, p. 163). There are instruments available to measure teacher self-efficacy, mathematics and science teaching efficacy, middle school statistics, and chemistry teaching self-efficacy. However, there is a need for a more accurate measure of teacher self-efficacy that is content and concept specific with respect to algebra (Hillman, 1986; Usher & Pajares, 2008).

This need is particularly important since there is a connection between teacher self-efficacy measures and student outcomes. Woolfolk-Hoy, Davis, and Pape (2006) and Woolfolk and Hoy (1990) noted that teachers’ sense of self-efficacy is one of the few teacher characteristics that has been consistently linked to students’ behavior and learning. Teachers’ self-efficacy is a crucial component in students’ achievement (Bandura, 1997; Tschannen-Moran & Woolfolk-Hoy, 2001; Tschannen-Moran, Woolfolk-Hoy, & Hoy, 1998), motivation (Midgley, Feldlaufer, & Eccles, 1989) and personal sense of academic efficacy (Anderson, Greene, & Loewen, 1988).

Conceptual Model

While there has been a focus on algebra standards for students (NGA & CCSSO, 2010), there has been little attention on the content knowledge and instructional strategies teachers must possess to help all students be successful in algebra. Research studies that have analyzed instructional strategies of teachers have often been general in application and not specific to the content the teacher is addressing (Balfanz, Legters, & Jordan, 2004; Williams, Haertel, Kirst, Rosin, & Perry, 2011). McCrory, Floden, Ferrini-Mundy, Reckas and Senk (2012), who developed the Knowledge of Algebra for Teachers (KAT), wrote that "tools are needed that measure the types of mathematical knowledge thought to be useful for teaching" (p. 610). The key being that there are different types of knowledge needed, including knowledge of school algebra and algebra-for-teaching knowledge. The question arises as to what that knowledge is and how one would determine that knowledge base. To that end the authors analyzed specific curriculum documents and literature relative to knowledge of school algebra and algebra-for-teaching knowledge.

The system for choosing and analyzing curriculum documents and literature included aligning both student and teacher standards for algebra. For students, curriculum documents included those that were utilized by the states and policy makers such as the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) and the *Common Core State Standards* (NGA & CCSSO, 2010). For teachers, curriculum documents included those utilized by teacher preparation programs, states, and policy makers such as the *NCTM Standards for the National Council for Accreditation of Teacher Education* (NCTM & NCATE, 2012) and *Conference Board of Mathematical Science Standards* (CBMS, 2012).

As a result of the systematic analysis of curriculum documents and literature, we conceptualized two main domains of efficacy: *efficacy of school algebra* and *efficacy of teaching school algebra*. These domains parallel recently defined dimensions by McCrory et al. (2012) with respect to knowledge of algebra for teaching - *knowledge of school algebra* and *teaching knowledge* respectively. Efficacy of school algebra refers to the ability "to do" the algebra taught in school mathematics. By the ability "to do" school algebra we mean the belief about his or her ability to meet the expectations prescribed by student and teacher standards, which include the knowledge and skills related to particular topics and processes. Efficacy of teaching school algebra refers to the ability "to teach" the algebra taught in school mathematics. By the ability to teach school algebra we mean the ability to teach others the algebra knowledge and skills expected to be learned in school mathematics. To further capture these ideas, we organized the topics and processes of school algebra into six categories - four of the categories are content topics (variables, functions, patterns, and modeling) and two are processes (technology and multiple representations).

Figure 1 is the conceptual model derived from existing theories about teacher knowledge, and teacher self-efficacy, which provides an overview of the factors that determine student outcomes focusing on teacher characteristics. The middle portion of the model consists of two ovals depicting two separate, but related, aspects of self-efficacy. They refer to the measure of the belief of teachers' own ability to do and to teach school algebra respectively. Both aspects of self-efficacy figure prominently as a direct predictor of student outcomes. As noted above, previous studies have found connections between teachers' sense of self-efficacy and student outcomes, hence a specialized self-efficacy should also be a predictor. Teacher knowledge of algebra is positioned at the top of figure 1 as a direct predictor of self-efficacy. We argue that when teachers possess subject matter knowledge and PCK, that is, MKT, they hold a belief of self-efficacy to do and to teach school algebra. As teachers acquire knowledge, through formal teacher preparation, professional development programs, and practice, it is hypothesized that their beliefs about their abilities change.

In order to empirically investigate the claims about the relationships between the three major aspects of the model, teacher knowledge of algebra, self-efficacy, and student outcomes we need direct measures that will give us an approximation of their quantification. These direct measures are depicted in figure 1 as rectangles. For *Teacher Knowledge of Algebra*, we propose the KAT measures. These are valid measures aligned with MKT and PCK theories specifically for algebra. For *Self-Efficacy* to do and to teach school algebra, no measures exist at the moment. However, we believe that, if these measures were created, they would need to be based on documents that delineate what is taught in school algebra such as student and teacher standards. In addition, they need to be informed by previous general education self-efficacy measures, and in particular self-efficacy measures in specific content areas such as Statistics and general mathematics. For *Student Outcomes*, we propose both cognitive and affective measures such as test scores, attitudes and motivation measures.

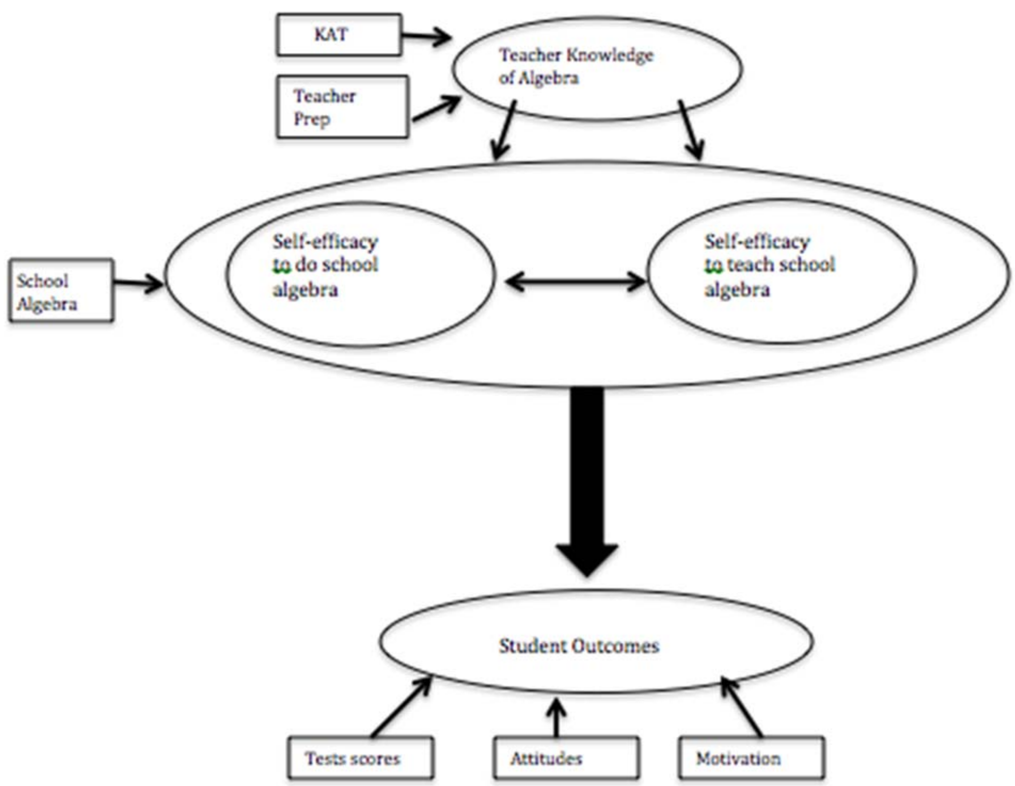

Figure 1. Conceptual model for algebra teacher self-efficacy.

Conclusion

The conceptual model is the precursor to measuring teacher self-efficacy to do school algebra, as well as teacher self-efficacy to teach school algebra. Both aspects are essential to improve the teaching of algebra. The development of an instrument based on this conceptual model has the potential to serve as a better evaluative tool for projects and programs related to teacher effectiveness when teaching algebra. Professional development initiatives for teachers require measuring their impact on student achievement in a particular content area. This conceptual model and its potential uses offer a foundation to provide teacher educators, professional development programmers, school level administrators, and state mathematics leaders with a tool to measure and assess the impact of efforts related to student algebraic understanding. Prior to developing research and assessment tools to measure teacher self-efficacy in algebra, the appropriate aspects of algebra content and pedagogy must be addressed.

The challenge for researchers - in both conceptual and empirical terms- – is to fill in the chains (the arrows in Figure 1) linking content knowledge and self-efficacy measures with student outcomes.

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RELATIONSHIP BETWEEN COGNITIVE TYPES OF TEACHER CONTENT KNOWLEDGE AND TEACHING EXPERIENCE: QUANTITATIVE STUDY OF MEXICAN BORDERLAND MIDDLE SCHOOL TEACHERS

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This ongoing mixed methods study will provide understanding of the relationship between mathematical teacher content knowledge and teaching experience of Mexican middle school teachers. In this paper, a correlation analysis will be presented which is part of the quantitative phase of the entire study. Teachers (n=55) from grades 7-9 completed the Teacher Content Knowledge Survey (TCKS) for data collection. An examination of each cognitive type of content knowledge and the overall TCKS score as they related to years of teaching experience, and the correlations among the cognitive types of teacher content knowledge is presented.

Problematizing the Issue

In the past twenty-five years, there are number of studies that have focused on teacher knowledge (e.g. Shulman, 1986; Tchoshanov, 2011). However, teacher knowledge is very broad, and it includes different kinds of knowledge. In mathematics, scholars have addressed kinds of knowledge and their components (e.g., An, Kulm, & Wu, 2004; Tchoshanov, 2011). The criticism and classification of the different types of knowledge that a teacher should possess in order to teach mathematics effectively are relevant. Some of the categorizations of teacher knowledge in mathematics are teacher content knowledge (e.g. Tchoshanov, 2011), pedagogical content knowledge (e.g. An et al., 2004), knowledge of curriculum (e.g. Ball, Thames, & Phelps, 2008; Shulman, 1986), knowing-to act (e.g. Mason, 1998), among others. The complex nature of the mathematical knowledge for teaching challenges scholars to research and define with precision each kind of teacher knowledge. In addition, interactions among these kinds of knowledge are crucial as a part of the knowledge base for teaching mathematics. Therefore, further research is needed on the interaction among specific kinds of teacher knowledge. Additionally, to know what kinds of knowledge persist or improve through the years of teaching experience is relevant to understand the teaching practices in mathematics classrooms.

The intent of this study is to measure middle school mathematics teachers' content knowledge in Mexico. Teacher mathematical content knowledge includes three different cognitive types: knowledge of facts and procedures (we will further refer to it as T1); knowledge of concepts and connections (T2); and knowledge of models and generalizations (T3). In this paper, we examine these cognitive types and seek the relationship among them and teaching experience. In addition, correlations are analyzed to understand how teacher mathematical knowledge associations are composed. In order to achieve our purpose, a survey was administered to collect data from 55 middle school mathematics teachers in the Mexican borderland.

Educational research in Mexico is not frequently promoted due to the lack of research funding (Reyes, 2013). There is a dearth of studies in Mexico that focus on teacher content knowledge and its' relationship with teaching practices at the middle school level. For example, Castañeda, Rosas, and Molina (2011) focused on the mathematics discourse used by mathematics middle school teachers to formulate generalizations, synthesis and summaries of classroom activities. Dueñas (2009) studied the social construction of the nature of mathematics teaching and learning done by middle school pre-service teachers. Mochon and Hernandez's study (2011) analyzed the content knowledge and perspective of mathematics middle school pre-service teachers at the end of the teacher preparation through a special final course. Inzunsa and Guzman (2011) analyzed the teachers' understanding of probability concepts. Also, Mochon and Andrade (2009) studied the arithmetic knowledge of elementary school teachers. Thus, the existent lacuna of research on mathematical teacher content knowledge in Mexico is evident. Further research that focuses on teacher mathematical content knowledge at the middle school level in Mexico is needed.

Conducting research in Mexico will allow making comparison studies with other countries such as the United States where studies about teacher content knowledge have been widely addressed (e.g. Tchoshanov, 2011; Ball et al., 2008; Hill, Ball, & Schilling, 2008). However, most of those studies include participants within the United States and few other European countries. The insights of this investigation intend to contribute to the field of inquiry related to mathematics teacher knowledge with participants selected from Mexican middle schools.

This research will contribute to teacher preparation programs by adding relevant information to the educational policy makers. Information that will inform about what type of knowledge has a stronger impact on teaching is presented in this paper. The research questions addressed in this paper are: (a) to what extent is the cognitive type of Mexican middle school mathematics teachers' knowledge associated with teaching experience? (b) Is there a significant correlation among the cognitive types of Mexican teachers' content knowledge?

Conceptual Framework

In this paper, we focus on content knowledge of teachers and how it is associated with teaching experience. Based on Shulman's (1986) categories of teacher knowledge, we consider teacher content knowledge as "the amount and organization of knowledge per se in the mind of teachers" (Shulman, 1986, p. 9). It includes the knowledge "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al., 2008, p.377-378). Hence, three cognitive types of mathematical teacher content knowledge were identified: cognitive type T1-knowledge of facts and procedures; cognitive type T2-knowledge of connections and concepts; and cognitive type T3-knowledge of generalizations and models (Tchoshanov, 2011). This study analyzes these cognitive types of teacher content knowledge through data collection of total scores that teachers gain in the survey. Memorization and application of basic mathematical facts, rules, and algorithms to solve routine problems is required for type 1 knowledge (Skemp, 1978; Stein, Smith, Henningsen, & Silver, 2000). The quantity and quality of connections between mathematical procedures and ideas are part of the mathematical conceptual understanding in which the cognitive type 2 is focused (Tchoshanov, 2011). Type 3 knowledge focuses on the theoretical part and includes conjecturing, generalizing, proving theorems, etc.

Methodology

Participants

The sample consisted of 55 Mexican middle school teachers. However, since the study is in progress, in this paper the analysis of 43 teachers' content knowledge is presented. Participating teachers are currently teaching in middle schools in Mexico with teaching assignment of one or more courses of mathematics. In Mexico, the middle school consists of the grade levels from 7 to 9. Of the total participants, 18 (out of 43 surveyed teachers) were female teachers, and the rest were male teachers. This sample was composed by 65% of teachers who have more than 16 years of teaching experience. Most participants (65%) teach a particular grade level, however, some teachers teach other courses not related to mathematics. Sampling selection used in the study is a convenience sampling (Tashakkori & Teddlie, 2003). Mexican education authorities provided access to only one of the middle school subsystems in Mexico. Therefore, Mexican teachers who work in this subsystem were voluntarily selected to participate in this study.

Research Design

A mixed methods sequential explanatory design will be utilized in the ongoing study. This design implies the sequential implementation of a quantitative phase followed by a qualitative phase. In the quantitative phase, numerical data will be collected through the administration of two surveys: 1) teacher content knowledge survey (TCKS); and 2) knowing-to act survey (KtAS). Then, the analysis of the answers of these surveys will be performed in order to make a case selection (N=4) and develop an observation and interview protocol. After that, classroom observations and interviews will be implemented. Using the results of the analysis of the qualitative data, we will interpret and explain the results obtained in the quantitative phase. However, only a part of the quantitative phase is presented in this paper.

TCKS Instrument

The instrument used to measure teachers' mathematical content knowledge through its three cognitive types is the Teacher Content Knowledge Survey (Tchoshanov, 2011). This survey is composed of 33 multiple choice-items about relevant topics for middle school grades teachers' knowledge: Number Sense, Algebra, Geometry and Measurement, Probability and Statistics. 10 items measured the cognitive type 1. There were 13 items that assessed the cognitive type 2. And the rest of the 10 items assessed the cognitive type 3. There is no identification of the items with regard to cognitive type in the teacher content knowledge survey. Items of the three different cognitive types are located randomly throughout the survey. For instance, the item 7 measures the T1, the item 8 examines the T2 and the item 9 measures T3 as observed in table 1.

Table 1

Example of TCKS Items

Item	Answer options
(7) What is the rule for fraction division?	A. $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ B. $\frac{a}{b} \div \frac{c}{d} = \frac{ab}{cd}$ C. $\frac{a}{b} \div \frac{c}{d} = \frac{cd}{ab}$ D. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$
(8) Which of the following problems represents the operation below? $1\frac{3}{4} \div \frac{1}{2} = ?$	A. Juan has a piece of rope $1\frac{3}{4}$ feet long and cuts it in half. At what length should he cut the rope? B. Maria has $1\frac{3}{4}$ liters of juice. How many $\frac{1}{2}$ liter containers can she fill? C. A boat in a river moves $1\frac{3}{4}$ miles in 2 hours.

	What is the boat's speed? D. Daniel divides $1\frac{3}{4}$ pounds of coffee evenly between 2 customers. How many pounds of coffee will each customer get?
(9) Is $\frac{a}{b} \div \frac{c}{d} = \frac{ac}{bd}$ ever true?	A. Always true B. Sometimes true C. Never true D. Not enough information to tell

An interdisciplinary faculty with expertise in the following domains: mathematics, mathematics education, statistics and statistics education, developed the instrument. This faculty represents various institutions such as university, community college and local schools. The main steps were the selection of items for the survey, the classification of items by cognitive type, and modification of items for other cognitive types. During 2005-06, the TCKS instrument was field-tested (Tchoshanov, 2011). In order to evaluate the reliability of the teacher content knowledge survey instrument, the alpha coefficient technique (Cronbach, 1951) was used. "The value of the coefficient of .839 suggests that the items comprising the TCKS are internally consistent" (Tchoshanov, 2011, p.148).

Data Collection and Analysis

This study is in progress. The data collection for the quantitative phase is almost finalized. From 55 mathematics middle school teachers that will be participating in the study, 43 teachers have completed the two surveys. The content knowledge survey (TCKS) was administered to N=43 teachers. Teaching experience is a variable that is important to analyze in order to observe how the teacher content knowledge is changing or immutable through the years of teaching mathematics. Thus, analysis of variance (ANOVA) is conducted to understand the relationship between teacher content knowledge (total scores and each cognitive type) and years of teaching experience. In addition, we analyzed the correlations among the different cognitive types of teachers' mathematical content knowledge. Different cognitive types of teacher knowledge (T1, T2, and T3) are examined. Indeed, this research analyzed the correlation between each cognitive type of teacher content knowledge to deepen the understanding of mathematical content associations. The first correlation sought was cognitive type1 and cognitive type 2, then, cognitive type 1 and 3, and finally cognitive type 2 and 3.

Results and Discussion

The following data are representative of the results obtained in this study, which examined the relationship between cognitive types of teacher content knowledge (T1, T2, T3, and total score of the TCKS) and years of teaching experience. Results of the study show a statistically significant correlation between cognitive type T1 of teacher content knowledge and years of teaching experience: $r(43)=.45, p<.01$. This correlation informs that teacher knowledge of mathematical facts and procedures is more solid through the years of teaching mathematics. Hence, Mexican teachers who have more years of teaching experience possess stronger knowledge of basic facts, algorithms, and procedures.

The second substantial finding was the correlation between teachers' total score on TCKS and teaching experience presented in Table 1. The analysis of variance showed a significant correlation between teacher content knowledge measured as the total score on the TCKS and years of teaching experience (Pearson's $r(43)=.32, p< .05$). The corresponding F value was $F(1,41)=4.840, p<.05$. In other words, the teachers' performance on the TCKS is significantly related to teaching experience, indicating that teachers who performed better in the content test overall, are those who have more years of teaching mathematics.

Table 1: Relationship between teachers' mathematical content knowledge and years of teaching experience

ANOVA	df	SS	MS	F	Significance F
Regression	1	347.317042	347.317042	4.84049809	0.033491126
Residual	41	2941.84575	71.7523353		
Total	42	3289.16279			

The correlational analysis between cognitive types T2 and T3, and teaching experience is not significant (for T2 Pearson's $r(43)=.22, p>.05$; for T3 $r(43)=.10, p>.05$). However, it is interesting to notice that total scores of TCKS including the three cognitive types are strongly related to the years of teaching experience as it is shown in table 1. Therefore, we add to the discussion that teachers with more years of teaching experience, overall, tend to be more knowledgeable in mathematics content.

The results of three correlational analyses show interesting relationships. The first correlational analysis examined T1 and T2. The results for this comparison reported a strong correlation between T1 and T2 types of mathematical content knowledge ($r(43)=.55, p<.01$).

Therefore, we can deduce that a teacher who possesses the knowledge of concepts and connections has a comprehensive foundational knowledge of facts and procedures. The second analysis focused on comparing T1 and T3. It showed significance (*r(43)=.55, p<.01*). This result infers that a teacher who is able to do mathematical models and generalizations is also knowledgeable of procedures and facts of mathematical content.

Finally, an unexpected finding is reported. There is no substantial significance showed in the correlation between cognitive types T2 and T3 (*r(43)=.24, p>.10*). Thus, the fact that a teacher possesses conceptual knowledge that enables to make connection among mathematical concepts does not indicate that the teacher knows mathematical models and generalizations. We can say that cognitive type T1 of teacher content knowledge is the foundational knowledge for teachers to be able to understand mathematical concepts and make connections, and develop an understanding of mathematical models and generalizations.

Conclusions

The results of this study indicate that teaching experience associated with teachers' content knowledge. This result suggests that teacher knowledge of mathematical facts and procedures become more solid through the years of teaching mathematics. In addition, findings of the study highlight the importance of teacher knowledge of facts and procedures that serves as a foundation to construct conceptual and connected knowledge as well as make generalizations and apply mathematical models. Teachers with a strong mathematical content knowledge are able to make connections among mathematical concepts in order to help students to make sense of mathematics. Congruently, weak mathematical teacher content knowledge affects mathematical instruction, which, in turn, causes poor opportunities to learn as well as students' frustration and negative disposition toward mathematics (Sorto, Marshall, Luschel, & Carnoy, 2009).

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THREE-COLUMN PROOFS FOR ALGEBRAIC REASONING AND JUSTIFICATION

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This study implemented three-column proofs as a means to aid pre-service teachers (PSTs) in reasoning and justifying algebraically. Three-column proofs were implemented in a capstone mathematics course for secondary school PSTs focusing on algebra, functions, and probability. PSTs used three-column proofs within the course, and voluntarily participated in a ten-question survey discussing implementation of three-column proofs into secondary education. The results demonstrated that PSTs value three-column proofs above standard solve for x problems because students must explicitly justify why each step involved in solving an equation. PST survey results and PST performance will be discussed.

Over a decade ago, the National Council of Teachers of Mathematics (2000) argued that proof should be pervasive in secondary education, yet proof still revolves around the geometry curriculum. The Common Core State Standards (CCSSO, 2010) use the words proof, prove, or proving twice in Algebra Standards, three times in Function Standards, and ten times in the Geometry Standards. Knuth (2002) and Wu (1996) argue that by postponing proofs until secondary geometry, educators cannot expect students to perform intricate justifications. Wu (1996) points out how this offers students a false pretense as to what defines proof as supported by Healy and Hoyle's (2000) work with secondary-school students.

This paper introduces three-column proofs as a means to encourage algebraic reasoning and justification with preservice secondary teachers (PST) of mathematics. This study has PSTs discuss three-column proofs within their class and secondary school implementation. Examples of three-column proofs via PST classwork and PST survey responses are discussed.

Framework and Literature

Algebra classes are traditionally engrossed in having students "solve for *x*." Consider the traditional example:

Solve for x.

$$\begin{array}{l} 3x + 5 < 7 \\ -5 \qquad -5 \\ \frac{3x}{3} < \frac{2}{3} \\ x < 2/3 \end{array}$$

Figure 1. Example of secondary-school students' work in solving inequalities.

The CCSS (2010) state that with equations and inequalities, students should "Construct a viable argument to justify a solution method" (CCSS.Math.Content.HSA-REI.A.1). Figure 1 lacks a viable argument and justification. One may argue that this symbolic representation is justification, but many studies have shown that students have not internalized symbolic representations despite their popularity (Healy & Hoyles, 2000; Knuth, 2002; Yopp, 2011). Mathematics education lacks a generalizable means for students to express algebraic reasoning and justification when solving for *x* via a "viable argument."

The epistemological framework for algebraic reasoning and justification parallels Stylianides' (2007) perspective of "proof as a mathematical argument."(p. 291). Inspired by Fawcett's work (1939) in geometry, Stylianides (2007) emphasizes the need for reasoning and justification beyond correct symbolic representations or decontextualized abstraction and towards hermeneutic understanding in practice. Concurrently, Knuth's (2002) recognizes that proof in the secondary classroom stems from the teacher's belief in proof and that sadly it is still marginalized to geometry classroom. To change these beliefs that proof is compartmentalized and marginalized in the secondary mathematics curriculum, ontological reconceptualization must be made available for PSTs in their education. This study offers PSTs this ontological reconceptualization through an explicit structure to aid PSTs in understanding proof in algebra classrooms.

To help PSTs make viable arguments, three-column proofs (Table 1) were introduced in a capstone mathematics course to meaningfully demonstrate algebraic reasoning and justification. Table 1.

Example of three-column proofs

Statement	**WHAT you did**	**WHY you could do it**
$3x+5<7$	Given	Because it was given
$3x<2$	Subtracted 5 from both sides.	I can apply *h(x)=x-5* to both sides of the equation Since *h(x)* is a strictly increasing function the inequality is preserved, $3x+5<7 \Rightarrow h(3x+5)<h(7) \Rightarrow 3x<2$.
$x<\frac{2}{3}$	Divided both sides by 3.	I can apply *h(x)=x/3* to both sides of the equation Since *h(x)* is a strictly increasing function the inequality is preserved, $3x<2 \Rightarrow h(3x)<h(2) \Rightarrow x<\frac{2}{3}$.

In Table 1, the second column has PSTs explain their reasoning while the third column has PSTs justify their reasoning. This technique helped PSTs see how significant proof is to algebra and how to implement proof in the algebra classroom.

This model does satisfy Stylianides' (2007) tenets for proof in school mathematics: set of accepted statements, modes of argumentation, modes of argument representation (p. 291). Three-column proofs have the set of accepted statements as the third column normalized by the classroom community as properties, theorems, and justifications. The mode of argumentation is demonstrated by the first and second column's purpose to explain student reasoning. Finally, the three-column structure constitutes the representation for the mode of argumentation. Thus despite not indicating the result to be proved (only solved) such justification may indicate proof.

When looking to the history of implementing structure within proofs, there are many warnings to heed from the standardization of two-column proofs in geometry in the 1930's (Herbst, 2002). Specifically, Herbst (2002) suggests that any design should not be standardized as proof, but selectively implemented where meaningful reasoning and justification are useful. To this end, this study only includes preliminary data so as not to rush any structures without fully understanding the ramifications of their implementation. The challenge of three-column proofs to students as learners-of-proof (Knuth, 2002) would be too far-reaching for an initial study. Hence, this study began with senior undergraduate PSTs who have already taken multiple proof-based math classes and focused on two research questions.

1. How do three-column proofs help PST's learn about algebraic reasoning and justification?
2. How do PSTs perceive this could this be implemented in the secondary school classroom?

Method

To answer these research questions, eight senior undergraduate PSTs participated in a survey discussing three-column proofs in a capstone mathematics content course in California. The use of three-column proofs was fully integrated into the capstone course. Fundamentally, three-column proofs are applicable to any "solve for *x*" problem with an agreed upon structure and set of justifications clarified by the instructor. The PSTs used three-column proofs for many "solve for *x*" problems beyond the examples demonstrated here. PST feedback was collected via a survey and their performance on assessments. Six survey questions were relevant to the research questions.

Q5. Please write a short paragraph about the advantages and disadvantages of using 3-column proofs in class.

Q6. What did it helped you learn in particular? How did it help you learn in particular?

Q7. Would you consider this a mathematical proof for what is asked? Please explain.

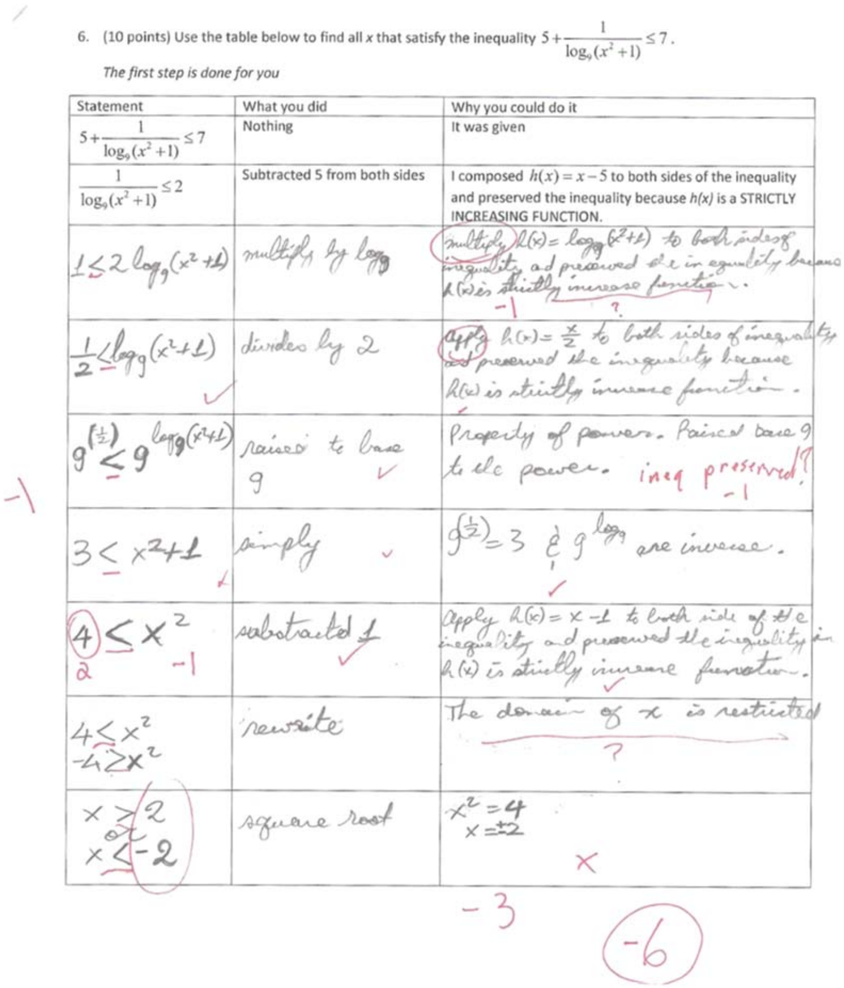
Q8. What aspects or properties of proof are valuable to you? What aspects of proof are found or missing in 3-column proofs?

Q9. Please write a short paragraph about the advantages and disadvantages of using this in Secondary School Algebra. How should it be modified for Secondary Schooling?

Q10. How would this help Secondary Students or Teachers in Algebra in particular?

Q5 through Q8 were designed to answer the first research question on PST understanding, while Q9 and Q10 were designed to answer the second research question about PST perceptions of secondary school implementation. Data from the survey questions were analyzed coding for common PST responses. Participants were labeled PST1 through PST8. Four PSTs were male and four PSTs were female.

One example of PST's work (PST5) from an exam (including the researcher's assessment, Figure 2) illustrate the significance and context of three-column proofs necessary for interpreting the survey results.


6. (10 points) Use the table below to find all x that satisfy the inequality $5+\frac{1}{\log_9(x^2+1)} \le 7$.

The first step is done for you

Statement	What you did	Why you could do it
$5+\frac{1}{\log_9(x^2+1)} \le 7$	Nothing	It was given
$\frac{1}{\log_9(x^2+1)} \le 2$	Subtracted 5 from both sides	I composed $h(x)=x-5$ to both sides of the inequality and preserved the inequality because $h(x)$ is a STRICTLY INCREASING FUNCTION.
$1 \le 2\log_9(x^2+1)$	multiply by $\log_9$	multiply $h(x)=\log_9(x^2+1)$ to both sides of inequality and preserved the inequality because $h(x)$ is strictly increase function. −1 ?
$\frac{1}{2} \le \log_9(x^2+1)$ ✓	divides by 2	apply $h(x)=\frac{x}{2}$ to both sides of inequality and preserved the inequality because $h(x)$ is strictly increase function.
$9^{(\frac{1}{2})} \le 9^{\log_9(x^2+1)}$	raised to base 9 ✓	Property of powers. Raised base 9 to the power. ineq preserved? −1
$3 \le x^2+1$ ✓	simply ✓	$9^{(\frac{1}{2})}=3$ & $9^{\log_9}$ are inverse. ✓
$4 \le x^2$ 2 −1	subtracted 1 ✓	Apply $h(x)=x-1$ to both side of the inequality and preserved the inequality because $h(x)$ is strictly increase function. ✓
$4 \le x^2$ $-4 \ge x^2$	rewrite	The domain of x is restricted ?
$x \ge 2$ or $x \le -2$	square root	$x^2=4$ $x=\pm 2$ ✗

−1 −3 −6

Figure 2. PST5's less-thorough work on Exam 2 including the researcher's assessment.

Figure 2 was chosen so that student work and assessment could be demonstrated together. The researcher was lenient in assessing the language of the third column as this was the first implementation of three-column proofs on an exam. When compared to higher-performing PSTs, PST5's third column lacks the language of other student's third column. PST1 uses justifying words such as "since" and "because" while these words are absent with PST5. It is difficult to determine if the lack of words indicating justification affected PST5's ability to ascertain understanding of proof or convince the reader of his understanding of proof (Stylianides, 2007).

Results

Many PSTs responded to the survey questions demonstrating a genuine interest in sharing their experience with three-column proofs. All eight surveys were coded summarizing the PST's response. For example, PST7 wrote the following when responding to Q10:

> This will help secondary students in Algebra by making them think about what they are doing instead of just copying the example that the teacher provided them with. This will help reduce the amount of mistakes they make. It will help the teacher identify where most students are struggling, and then the teacher knows what they said they should focus on test review days. They might have to make modifications next time they teach the lesson. (PST7)

From this response, PST7's responses were coded as "Can't copy, must understand" and "Reduce student mistakes by highlighting flaws." Additionally, PST7's response was coded as "Identify where students struggle,“ and "Helping teachers with review." PST's survey results are summarized in Table 2. *Summary of qualitative analysis of survey.*

Questions	**Responses (Number of Students out of 8)**
Q5. Please write a short paragraph about the advantages and disadvantages of using 3-column proofs in class.	ADVANTAGES • Helps understand why (6/8) • Helps one remember (1/8) • Aligns with geometric proof (1/8) • Helps students follow logic (1/8) DISADVANTAGES • Time Consuming (4/8) • Not enough provided space (2/8) • Grading justification (1/8) • No Disadvantages (1/8) • Too advanced (1/8)
Q6. What did it helped you learn in particular? How did it help you learn in particular?	• Understanding why (4/8) • Content knowledge (4/8) • Nothing (1/8) • Content is needed for teaching high school (1/8) • Knowledge needed for the next step (1/8) • The obvious step isn't necessarily the right step (1/8)
Q7. Would you consider this a mathematical proof for what is asked? Please explain.	• Yes (6/8) • No (1/8) • 50/50 (1/8) A Proof is: • Logical statements achieving a conclusion • Made by a statement • Step-by-step process • Logical explanation
Q8. What aspects or properties of proof are valuable to you? What aspects of proof are found or missing in 3-column proofs?	VALUABLE • Understanding why/theory behind proof (4/8) • Breaking proof down into smaller steps (1/8) • Reviewing mistakes on quizzes and tests (1/8) 3-COLUMN ISSUES • Repetitive (2/8) • Teacher can't vary problems (1/8) • Second Column isn't needed (1/8)
Q9. Please write a short paragraph about the advantages and disadvantages of using this in Secondary School Algebra. How should it be modified for Secondary Schooling?	ADVANTAGES • Algebraic Justification (8/8) MODIFICATIONS • Give first two columns, ask for third. • Scaffold 3-column proof. First leave a few blanks in a few columns and slowly add more blanks. • Combine second and third columns. DISADVANTAGES • Time consuming (2/8) • Students prefer to memorize steps (2/8) • Turn students off to math (2/8) • 3rd column too advanced (2/8) • Repetitive (1/8)
Q10. How would this help Secondary Students or Teachers in Algebra in	HELP STUDENTS • Can't copy, must understand why(3/8) HELP TEACHERS • Identify where students struggle (4/8) • Help teachers with review (1/8)

particular?	• Reduce student mistakes by highlighting flaws (3/8) • Stay organized (1/8)	• Improve teacher content knowledge (1/8) • Grading is easier (1/8)

To answer the research questions, common themes among PST responses were compared in Q5 through Q8 and then common themes were compared with Q9 and Q10.

In Q5 and Q6, justification (understanding why) was discussed with six out of eight and four out of eight of the students respectively. In Q7, six out of eight students considered three-column proofs formal mathematical proof, one out of eight of the students did not consider it formal mathematical proof, and one out of eight of the students considered it a half-proof. PST2 explained his reasoning of half-proof:

> I would say I'm 50-50 because it is a good technique for a proof, but most teachers right now are teaching students to memorize formulas and rules and not explain how and why….On the other hand you can try this and change the teaching of old ways. (PST2, Q7)

Nonetheless PST4 advocated, "I would definitely call this a proof. In fact, it is quite rigorous; justifying every step is similar to math 302 proofs [Modern Algebra]." (PST4, Q7) Once again, the student's use of the word justifying occurred frequently. Finally, Q8 also demonstrated the significant value in justification as four out of eight of the students stated that they valued understanding the "why" behind the proof.

Q9 and Q10 address how PSTs could see value of this in secondary education. In Q9, all students agreed that this is valuable because it requires algebraic justification. While the PST's did mention the weaknesses of three-column proofs in secondary schools (Q9: 2/8 too time consuming, 2/8 students prefer memorizing, 2/8 third column too advanced, 1/8 too repetitive), they also offered excellent suggestions for modifications so that implementation could succeed. PSTs suggested scaffolding the design where students begin by filling in a couple blanks within a three-column proof to slowly develop students' understanding of proof. The method of filling-in-the-blanks was also suggested for purposes of saving time.

PST responses to Q10 offered insight into how three-column proofs can help secondary students stay organized (2/8), highlight conceptual mistakes (3/8), and mandate understanding why (3/8). Moreover, half of the PSTs described three-column proofs as a way to help teachers with identifying struggling students in Q10. Thus Q9 and Q10 responses demonstrated that the overall strength of three-column proofs is the necessity of justification (similar to Q5, Q6, and

Q8), and while modifications are necessary for implementation at secondary schools, PSTs were optimistic about integrating such ideas.

Conclusions

Illuminating our first research question on PST learning, a majority of PSTs suggested that three-column proofs help in algebraic justification (Q5, Q6, Q8). Moreover, a majority of PSTs appreciate three-column proofs as actual proofs (Q7) despite lacking a formal statement to prove. PSTs did critique the design as being difficult for students, time consuming, and repetitive (Q5, Q8). These critiques must be considered for future use with PSTs.

The second research question was addressed by PSTs with valuable suggestions. All PSTs considered algebraic justification (Q9) an advantage worthy of secondary school implementation, while many suggested the third column would be challenging. Nonetheless, PSTs offered engaging options for teachers and students of secondary school (such as filling in some of the columns) to include three-column proofs in the secondary algebra curriculum. PST6 summarizes the relationship of three-column proofs with students concisely:

> Schools seem to teach math in a systematic way which overlooks the small details that actually turn out to be crucial. This 3 column proof exposes students to a level of thinking that no other method, I think, will help them do. Students are so use to the "how" but are rarely ever taught the why. (PST6, Q9).

Thus PSTs are optimistic about implementing three-column proofs in their secondary classroom to focus on student justification, but also are practical about its implementation.

Requiring justification within algebra classes is what the Common Corse State Standards encourage (CCSSO, 2010). However, research has suggested that clarity is needed or mathematics educators jeopardize blurring the lines between problem solving and problem proving (Yopp, 2011; Herbst, 2002). While many research studies (Healy & Hoyles, 2000; Knuth, 2002; Stylianides, 2007) have encouraged justification through interactive activities using generalization, few have offered a structure to “solve for *x*.” This study found three-column proofs could offer this structure while preserving justification so that proof is not thrust upon secondary students in geometry alone (Wu, 1996). This study found PSTs appreciated three-column proofs for their justification and offered strategies to improve its implementation.

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PLACING STUDENTS IN A MATHEMATICS COURSE: WHAT WORKS BEST?

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We consider various methods of placement of undergraduate first-time freshmen in mathematics courses. COMPASS, a computer-adaptive testing program, has been used at our university to help place students in appropriate courses for the past few years. Because it is costly and time-consuming to conduct it on campus at the time of student orientation, a proposal to use student SAT test scores instead was considered by our enrollment management. We conducted a comparative analysis of COMPASS versus SAT for placing students in suitable courses, insuring their subsequent success. The results of the statistical analysis are presented and discussed.

The procedure of placing a first-time freshman in an appropriate mathematics course is very important because it directly affects learning outcomes. If a student is incorrectly placed in a class that is too easy, the student is likely to spend valuable college time hardly learning anything new. If a student is placed in a class that is too hard, the student is likely to have a difficult time learning the material, and may be unable to grasp important concepts needed for further coursework. He or she may wind up repeating the class, failing again, and dropping out of college altogether.

A mathematics placement test developed by our department had been used for years at St. Mary's University, a Catholic four-year liberal arts college. Starting with 2008, our university decided to switch to using the COMPASS test, developed by ACT Inc. The test is administered to the majority of incoming freshmen during student orientation. Because the test is computer-based, it has to be administered in a computer lab while being proctored by the university staff. Since SAT/ACT scores are readily available for most of our students, the enrollment management office wanted to use those scores in place of COMPASS in order to place students in correct courses. Thus, the purpose of this study is to answer the following research questions:

- How well is COMPASS performing in placing our students in college-level math courses versus remedial courses?
- Are SAT math scores good predictors of student success in college-level math courses? Can they be used in most cases instead of COMPASS test for accurate placement decisions?

Related Research

Scott-Clayton (2012) conducted an analysis of predictive validity of COMPASS using traditional measures, such as correlations, as well as accuracy and success rates as recommended by ACT (2004). In addition to these measures, Scott-Clayton also proposed a new measure, which she calls a "severe error rate". Severe error rate combines students who are predicted to earn a B or better, but instead are placed into remediation, and students who were placed into college-level courses, but did not succeed. She found that even though COMPASS does have some predictive power, it is also associated with considerable "severe error rate". She pointed out that using high school transcript information may help make better placement decisions while keeping the percent of students who are placed into remediation the same.

Another study was conducted by Pansy Waycaster (2004), a professor of mathematics at Virginia Highlands community college. She wanted to see how well COMPASS, ASSET, and their own in-house readiness test were doing for placement of students into developmental algebra courses. The examination revealed that the COMPASS test was not a significant predictor of the final exam grades. She also discovered some inconsistencies between COMPASS and ASSET course placement recommendations. ASSET was the only placement test in her study that was found to be significant, and was further recommended as an alternative to using COMPASS for placement purposes.

Barr et al. (2002) has recommended in his report on COMPASS performance on placing students into English, mathematics, and ESL courses that the use of the test be either suspended altogether, or restricted to serving as only one of many tools used for advising. His conclusions were based on finding little or no relationship between COMPASS test scores and final grades.

Venezia and Voloch (2012) analyzed college placement exams in their study of college readiness for California State University (CSU). They found that about half of all students admitted to CSU in 2007, for example, required some type of remediation. The percentage of students needing remediation was even greater for community colleges. An Early Assessment Program (EAP) was implemented to help students learn about their level of college readiness. In addition to recommending that high school students take college placement tests early enough (around January), they also noted that these students must have an opportunity to take appropriate courses to bring them up to speed, and be ready in time for college course work.

Belfield and Crosta (2012) looked at high school GPA versus COMPASS or ACCUPLACER results as a possible predictor of success in college. They found that high school GPA seems to be a much better predictor of college GPA as compared to placement tests. However, because of concerns that high school GPAs are already somewhat inflated, caution should be exercised in relying on them too heavily. High school GPAs could lose their predictive power for admission and placement purposes (Sawyer (2013)).

Noble and Sawyer (2004) found that high school GPA scores and ACT composite scores were both effective in predicting first year college GPA of at most 3.0, but ACT was an effective predictor for all GPA levels.

Do students actually benefit from remediation if they are placed there? Some studies found that remediation may actually have an opposite effect for students whose scores are slightly below the cutoff placement score (Hughes and Scott-Clayton, 2011). This indicates that relaxing the cutoff scores necessary for admission into college-level classes may actually help with retention even if a slightly bigger percentage of students actually fail there.

Data Description

We have restricted the analysis to incoming freshmen from the School of Science, Engineering and Technology (SET) who needed to take a Precalculus class, which covers College Algebra and Trigonometry. We consider Precalculus to be the first college level course that these students need for their respective degree plans. Some students included in the analysis scored above the predetermined cutoff on the College Algebra portion of the COMPASS test, and were placed in Precalculus. However, about 40% to 50% for each year considered (2008-2012) did not meet the criteria for placement in Precalculus, so they were either placed in College Algebra, or even below that, in Intermediate Algebra or Math Skills.

Table 1: Breakdown of students taking Precalculus or remedial classes by year

Course	Percent Placed				
	2012	2011	2010	2009	2008
Intermediate Algebra or Math Skills	20.9%	30.2%	26.1%	21.2%	24%
College Algebra	20.3%	20.1%	14.2%	21.9%	20%
Precalculus	58.9%	49.7%	59.7%	57%	56%

For each student included in the analysis, their performance on COMPASS math test, SAT/ACT Math results, first mathematics course taken, and the final grade they received in it, were collected. Success in a class was defined as getting a grade of C or better, with D's, F's, W's and I's treated as failure.

Table 2: Failure Rates in remedial and college level classes by year

Course	Percent Failed				
	2012	2011	2010	2009	2008
Intermediate Algebra or Math Skills	21%	22%	26%	19%	25%
College Algebra	21.9%	33%	15.8%	15.2%	13.3%
Precalculus	13%	15%	18.8%	37.2%	16.7%

Analyzing Predictive Validity of Compass Math Placement Test Versus SAT Mathematics Test

Scores on COMPASS and SAT math tests are not directly comparable. Concordance charts exist between SAT and ACT scores, but there are no official such charts for COMPASS to SAT. The COMPASS test makers are not recommending these conversions because these tests may not have similar content, and are not administered in a similar manner. For example, COMPASS is taken on a computer, it is not timed, and is adaptive, so that students may not get to a certain level at all, and be done as quickly as half an hour. SAT is a paper-and-pencil timed test. SAT does not test trigonometry, so the test cannot be used for placement into Calculus I.

How do we compare scores on tests that are so fundamentally different? The traditional method of evaluating whether a particular test can be used for placement in a particular course is based on correlation coefficients. The correlations between the placement test scores and the final grades for Precalculus are shown in Table 3 below. If a placement test is working well, there should at least be a positive, significantly different from zero, correlation between the placement test score and the final grade for the course for which the placement test is being used. Generally, higher correlation coefficients indicate a better predictor of course performance.

However, since students who complete college-level courses are placed there based on their performance on the COMPASS test, there are no students with low COMPASS scores in these courses. Hence, the correlation coefficients between the COMPASS scores and the final grades in the corresponding college-level course (in our case, Precalculus) may be artificially low (see

ACT, 2004). We still report correlation coefficients as is the tradition in many research papers, but caution that their use can lead to incorrect conclusions about the validity of these tests.

Table 3: Correlations between SAT Math scores, COMPASS College Algebra scores, and final grades in Precalculus by year

Pairs of variables	Correlations				
	2012	2011	2010	2009	2008
SAT, COMPASS College Algebra	0.30	0.21	0.25	0.49*	0.05
SAT, Final Grade	0.40*	0.49*	0.40*	0.37*	0.23*
COMPASS College Algebra, Final Grade	0.13	0.13	0.28*	0.31*	0.10

Correlations marked with an asterisk * were significant at the 0.05 level.

Two statistics that measure predictive validity are success rate and accuracy rate (see, for example, Sawyer (2004)). These statistics rely on the fact that no placement test is perfect. There will be some students placed into the college-level course that should not have been placed there because they are underprepared (over-placement), and some students will be incorrectly placed into a remediation course, even though they would have succeeded in the college-level course (under-placement).

Table 4: Accuracy of Course Placement

Placement	Students succeed	Students do not succeed
Students are expected to succeed in a standard course	1. Correct decision	2. Incorrect decision (over-placement)
Students are not expected to succeed in a standard course	3. Incorrect decision (under-placement)	4. Correct decision

Thus, the success rate is the estimated percentage of students in cell 1 above, and the accuracy rate is the sum of estimated percentages of students in cells 1 and 4. Success rate, accuracy rate, and percentage of students who would be diverted to remediation, can help provide an optimal cutoff score that can be used with a given placement method.

Logistic regression is used to estimate the probability of success (defined for our case as passing a college-level class with a "C" or better), based on the score a student receives on his placement test. The logistic regression model with one predictor is given by

$$\ln\left(\frac{p}{1-p}\right) = \alpha + \beta x$$

where p is the probability of success in a standard course and x is the test score. The logistic regression intercept α and slope β are estimated from the students who took the standard college level course, and the probability of success for a given test score for each student who has it available is estimated by the equation

$$p = \frac{e^{\alpha+\beta x}}{1 + e^{\alpha+\beta x}}$$

For a given year, we fit a logistic regression model, where the explanatory variable was SAT math score or COMPASS College Algebra score, and the dependent variable was passing or failing the Precalculus class. For each year, the College Algebra score that students obtained from taking COMPASS was found not significant as a predictor of future success in the Precalculus class. For years 2010 to 2012, SAT math scores were found to be significant as a predictor, and were not significant for years 2009 and 2008.

Once a logistic regression model is fit, and found to have a significant predictor, we can estimate the probability of success in Precalculus given a student's test score, for all students who have such a test score available. The success rate is then the average probability of success for students above a given cutoff score. The accuracy rate averages the estimated probability of success for students above the cutoff and that of failure for students below the cutoff.

Since College Algebra COMPASS scores were not found to be a significant predictor, we only used logistic regression predictors for SAT math for years 2010, 2011, and 2012. The maximum accuracy rate is achieved when the estimated probability of success is equal to 0.5, and was found to be approximately 79% in 2012, 77% in 2011, and 73% in 2010, corresponding to a cutoff SAT math score of 460. This means that if a student scores at least 460, his or her estimated probability of success in the Precalculus course is at least 0.5. If a 75% probability of success (getting a "C" or better) is desired, then such a cutoff is estimated to be 520.

Discussion

Our analysis of five years of data (2008-2012) for incoming freshmen indicated that the COMPASS test was not found to be a good predictor of further performance in a Precalculus class. The correlations between the COMPASS College Algebra score and the final grade, even though positive, were not found to be significantly different from zero for the three out of the past five years. In addition, when the COMPASS College Algebra score was placed in a logistic regression model as a possible predictor of future success in Precalculus, the corresponding variable was not found to be statistically significant for every year included in the study. Thus, our findings indicate that the COMPASS placement test did not seem to be performing well for placing our students in Precalculus.

In contrast, we found that SAT math scores were a better predictor of success in a Precalculus class than the COMPASS College Algebra score. The correlations between SAT math score and the final grade in Precalculus were positive, and significantly different from zero for all five years incorporated in our research. Furthermore, when placed in a logistic regression model, SAT math was discovered to be a significant predictor of future performance in Precalculus for years 2010-2012. Logistic regression can be further used to justify SAT math cutoff scores for placing students into college level versus remedial classes.

The conclusion we make based on our findings is to exempt students from having to take the COMPASS test for placement into Precalculus if their SAT math score is at least 520. Students who score below 520 should take COMPASS in order to help with their placement. More research is needed to see how well COMPASS versus SAT math is performing in placing students in College Algebra versus Intermediate Algebra, and also Business Calculus versus College Algebra. However, COMPASS trigonometry scores should still be used for placement into Calculus because SAT math does not contain trigonometry.

We hope that our research so far, and the subsequent research we are planning to conduct, can help devise a better system for student placement. The initial steps a first-time freshman makes in college can be crucial for future performance and retention. A flawed placement policy may discourage a student from learning, and potentially result in delayed graduation or dropping out of college altogether. A considerable effort should be made to devise smart placement decisions, and minimize inevitable errors in student placement.

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WHAT'S A GOOD WAGER? COORDINATING STUDENTS' SURPRISING SOLUTIONS

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Using a contextual problem, I wanted, as a teacher, to see and hear how students generated their own solutions. The students' surprising solutions created two areas of investigation as a researcher. First, how does a teacher keep productive mathematical classroom conversations going when faced with unexpected and fascinating solutions worth additional exploration? Second, what is the combination of mathematical and pedagogical knowledge a teacher employs to keep worthwhile discussion moving and focused on mathematics? This paper will seek to provide answers to both questions.

As teachers, we face many opportunities where a student poses a challenging question to us. By challenging questions, I mean those classroom moments that push our own limits of knowledge of the subject in general and the concept in particular, as teachers and long-time students of the subject. What does a teacher do? How does a teacher respond? Although I have answered these questions in another work (Fox, 2011), I did so observing another teacher. In this particular report, I want to examine the same questions using myself as the research subject. The episode in this report is part of a larger study, reflecting on the blend of mathematical and pedagogical knowledge useful for a teacher to support regular classroom discussions with students.

Theoretical Framework

Within this particular activity, I had to do a lot, as the cliché goes, of thinking on my feet. Established literature—along with my previous work—provides the foundation for the work completed in this particular report. The basis for the original study (Fox, 2011) can be found in the work of Fernandez (1997). In the report for her study, Fernandez found that teachers in her study provided one of four responses when faced with a student's unanticipated question or comment: posing counterexamples, posing simpler or related questions, following through with the student's comment or question, and understanding/incorporating the comment into the classroom discussion. In implementing Fernandez's codes into subsequent studies, I have replaced the last code with an idea found in a report from science education, acknowledging challenging questions, as described in Park and Oliver (2009). In their report, Park and Oliver identified questions posed by intellectually curious science students for which the teacher could not provide an immediate answer. Upon the conclusion of the class, the teachers in Park and Oliver's study would use the students' question as an opportunity to research the answer to the student's question. In a later class session, the teacher would return to discuss the findings of the teacher's research.

Examining the work of teachers, the act of thinking on one's feet comprises part of the contingency component of the Knowledge Quartet, as proposed by Rowland and colleagues (Rowland, Huckstep, & Thwaites, 2005; Turner & Rowland, 2012). Turner and Rowland (2012) classified these classroom activities within the contingency component: “responding to children’s ideas; use of opportunities; deviation from agenda; [and] teacher insight” (p. 201). As a teacher presents a lesson, students provide input—either in the form of comments or questions—that cannot be anticipated, no matter the planning done on the teacher’s part. This component identifies those special moments in a classroom that encourages teachers to expect the unexpected. These codes were developed through observing novice elementary school teachers (Rowland, Huckstep, & Thwaites, 2005), but these codes could be applied to a variety of teachers in a variety of classroom settings.

Methodology

Separating this project from the original study is the use of action research in the current study. For both studies, the research questions are identical: “In what ways does a teacher apply a specialized mathematical knowledge for teaching when presented with an unanticipated student question? [and] What are the approaches a teacher uses when responding to a student who has posed an unanticipated question?” (Fox, 2011, p. 8). Engaging in action research allows the researcher to serve the simultaneous role of teacher. In this line of inquiry, one can provide an answer to the question posed by Schoen (1983): “How [are] professional knowing like and unlike the kinds of knowledge presented in academic textbooks, scientific papers, and learned journals?” (p. viii). To answer this question, an action researcher can use the statement from Cochran-Smith and Lytle (1993) as a response: “By inquiry, we suggest that teacher research stems from or generates questions and reflects teachers’ desires to makes sense of their experiences—to adapt a learning stance or openness toward classroom life.” (p. 24). Further justifying the rationale for possessing simultaneous roles of teacher and researcher, Cochran-Smith (2005) noted in her work as an action researcher “that there were not distinct moments when we were only researchers or only practitioners and thus to emphasize the blurring rather than dividing of analysis and action, inquiry and experience, theorizing and doing in teacher education” (p. 219). By serving the dual roles, I felt that I could gain a greater sense of what moments were truly unanticipated and could recall the types of knowledge I accessed when a student posed an intellectually challenging question or comment.

To support my work, I recorded all of the classroom discussions using two audio recording devices. I placed one recording device near where I stood to focus on my part of the classroom discussions. I placed a second device near the students to capture the discussion among the students. In the moments a student made a comment I did not anticipate, I would ask the student to provide additional information to his or her original response, and then provide the reasoning for the particular response. In these unplanned moments, I asked the student—and the whole class—to explain by asking the question like, “What do y’all think?”, and then having the student elaborate on that response.

To support the analysis of the data, I used two sources: reflective journal entries and lesson graphs. At the end of each day of the study, I wrote a brief entry in a reflective journal to support the initial analysis of the data. Like the original study (Fox, 2011), I used a lesson graph to support the analysis of this classroom episode. A selection of this episode's lesson graph can be found in Appendix B. The three columns of the graph represent the elapsed time in the episode, a note of the classroom discussion, and my initial analysis of the episode. From the initial analysis and my own reflections of the episode, I proposed answers to the questions found in the introduction of this report.

Findings

I had found myself having to acknowledge students' challenging questions—or even comments. The questions used in the activity can be found in Appendix A. Preparing for this activity from a content perspective, I supported my own answers and explanations using Gilbert and Hatcher's (1994) paper. Because the questions came from real-world settings, I also had available a second set of answers: the responses from the actual contestants. Only in a few of these situations did the two solutions—theoretical vs. actual—differ. I anticipated that students might provide different answers to my own, but—because I had my own justifications and convinced myself of those justifications—I could determine exactly what those responses and justifications might be. I will present the students' comments and questions in the order of the three scenarios in Appendix A. Focusing on a response to the second research question, I could identify my own responses to the students' unanticipated comments and questions through each scenario by broad themes: establishing, surprising, and exploring.

I established norms with the students while simultaneously seeking the students' responses to the first scenario in Appendix A. For example, the first student to offer me a response was for the first contestant to wager $400. I was a little surprised by the response, and offered the comment back to the student and the entire class, "I want to come back to that thought in a moment." Following that statement, four other students suggested their wagers for the first contestant: $2,000; $2,100; $3,000; and $2,199. Hearing these responses, I made statements to encourage greater discussion toward my prepared answers. Some students believed initially contestants had to wager in denominations of hundreds of dollars, instead of any whole-number amount less than or equal to the contestant's score. Another consideration was my realization that I may not have used precise language: by talking about an ideal wager, some students believed that so long as the first-place contestant won the game at the end of the final question, then the any wager less than the wager I had prepared would be an ideal solution. When a student made the comment, "I think everyone is right between those, um, between 2199 and 400, because [the contestant] would still win", I praised the student's comment highly, because of the validity of the statement, however I also wanted to focus the responses to a particular solution.

In the second scenario, I was surprised by the challenges the students faced during the discussion. The students' first challenge was not related to mathematics. After several minutes of discussion not related to mathematics, the students returned to the topic at hand and provide some fascinating discussion. One student quickly provides the answer and explanation for the first contestant that I had planned for; little discussion ensued. For responses to the second contestant in this scenario, I noticed an interesting deviation from my intended plan, illustrating a contingency moment from the Knowledge Quartet (Turner & Rowland, 2012). Although one student provided the answer and explanation I prepared for the second contestant, two students wanted to present a different solution and explanation to the same scenario. I gathered from the students that the student was satisfied in each contestant finishing in the same position as where the contestant started: their suggestion wager—a small amount—did not permit the second-place contestant from finishing in third place, but did not necessarily consider an optimal strategy for adding to the contestant's score. I believed that the student did not anticipate any contestant to answer the question correctly, and had developed a winning-by-not-losing strategy. The most surprising moment in this discussion occurred when one student wanted to handle multiple contingencies in a single wager, by suggesting several wagers for the same contestant. This student wanted to combine the two strategies mentioned in class into a single response. I wanted to permit the student as much as needed to communicate his thoughts clearly, but because of the struggle to articulate an explanation or question effectively, I decided to end the discussion at that point, to quell some of the frustration beginning to grow in the student—and even myself.

Having established and been surprised by the students' comments, the discussion of the third scenario allowed me the opportunity to explore the answers and explorations from the entire class. The students quickly provided the correct answer and corresponding explanation to the first contestant's wager. Although other students posed different initial responses, they did not vary greatly to the original response. When discussing the responses for the second contestant, I was surprised that the students suggested two responses that seemed valid in the moment of the discussion. The first approach the students suggested would be the one validated by the work of Gilbert and Hatcher (1994): the entire amount. The second—unexpected to me—approach was for the second contestant to wager enough to be greater than twice the third contestant's greatest possible outcome. Encouraging the student to explain the solution in greater detail, I became convinced of the validity of the solution and explanation. For the third contestant, I was expecting the students to suggest the entire amount. What I did not expect was one student taking the approach suggested by Gilbert and Hatcher. I had prepared to lead a mini-lecture on the reasoning associated with the response: a winning-by-not losing strategy where the third-place contestant finishes ahead of the other two contestants by having the least amount subtracted when all three contestants get the question wrong. Rather than telling the students the approach, I invoked a following through with the student's comment (Fernandez, 1997) to conclude discussion on this particular activity.

As a teacher, I still have ways to extend and improve the activity. Exploring this episode as a mathematics educator, I still have several opportunities to explore the ramifications of the pedagogical opportunities found in this episode. One takeaway I have serving the dual roles of teacher and researcher is that I feel confident addressing the latter of the two research questions in this study, but not as much addressing the former. Having an observer might be a reason why other reports (Fernandez, 1997; Fox, 2011; Park & Oliver, 2009) readily identified a specialized knowledge for teaching mathematics. An additional consideration is due to the nature of this lesson. While addressing a contextual problem seemingly without rigid solutions, I wonder to what extent would the opportunities exist for students to pose intellectually challenging questions and comments to the teacher or to other students? Extending the work presented here, this report focuses on how one class of students responded to the activity described in Appendix A. I believe the same activity could be performed in a variety of secondary mathematics classes. Additional research could focus on the types of responses those students provide and compare and contrast those responses to the responses anticipated by the teacher or by students in this report.

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Appendix A

List of Questions Used in Activity

Note: In this activity, the student is going to select all three wagers for the final questions for three "contestants" on the game show. If the contestant answers the final question correctly, the contestant wins (adds) the amount of the wager to his/her score. If the contestant answers the final questions answers the final question incorrectly, the contestant loses (subtracts) the amount of the wager from his/her score.

1. [Contestant A] has 17,400. [Contestant B] has 4,200. [Contestant C] has 7,600. What should each contestant wager for the final question? Why did you make each of the three choices?
2. [Contestant A] has 14,200. [Contestant B] has 6,400. [Contestant C] has 10,200. What should each contestant wager for the final question? Why did you make each of the three choices?
3. [Contestant A] has 19,600. [Contestant B] has 15,000. [Contestant C] has 14,800. What should each contestant wager for the final question? Why did you make each of the three choices?

Appendix B

Selection from Lesson Graph

Time	Noteworthy Classroom Moment	Initial Analysis of Moment
57:08	Explaining the "technical answers"	When I refer to the technical answers, I mean the answers I found by using the approaches found in the paper.
57:37	Showing the actual [Contestant] wagers everything, not the students' response of a smaller amount.	I found myself believing the students' response is right.
59:21	Showing the right response to the third-place contestant.	
59:45	Student tries to explain the newly presented solution is the most ideal strategy.	The first student doesn't complete the explanation, a second student does.
1:00:07	Explaining the "winning-by-losing" strategy in the second situation.	This approach actually foreshadows the next scenario.
1:01:46	Trying to explain a situation where the final score 1-0-0.	A lot of contingencies abound here, so I wonder if the students are trying to over-analyze the situation.

CHALLENGES OF USING VIRTUAL MANIPULATIVE SOFTWARE TO EXPLORE MATHEMATICAL CONCEPTS

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This study investigates usability challenges experienced by children working with virtual manipulatives. For the study, the way a second grade student manipulated virtual manipulatives was observed. Interview questions were used to elicit the participant's thinking during the various tasks. Using a laptop controlled by a computer mouse, the participant was asked to solve addition problems using the Base Ten Blocks activity from the National Library of Virtual Manipulatives. Observations of the student's manipulations of the on-screen blocks and a video recording of the computer screen were analyzed. This paper will share findings from the observations and discuss implications for the design of future virtual manipulative environments.

Theoretical Framework

It is often argued that mathematics is difficult to teach and learn since many math concepts are abstract in nature. Fortunately, many researchers and practitioners have generated creative solutions to help young children experience these abstract concepts more concretely. Perhaps the most well-known example of such an innovation is the introduction of physical manipulatives into mathematics education. Manipulatives are physical objects specifically designed to foster learning (Zuckerman, Arida, & Resnick, 2005). Examples of mathematical manipulatives include Tangrams, Cuisenaire rods, Numicon patterns, and Diene's blocks.

With the advent of technology, many researchers have started to study how physical manipulatives can be implemented, modified, or improved for digital environments. This has resulted in various forms of technologically infused manipulatives sometimes called computer-based manipulatives, digital manipulatives, or tangible interfaces. While theses learning tools are labeled differently, they are all essentially a "new class of manipulatives, called *virtual manipulatives*" (Moyer, Bolyard, & Spikell, 2002, p. 372). Moyer and colleagues define virtual manipulatives as "interactive web-based visual representations of dynamic objects that present opportunities for constructing mathematical knowledge" (p. 373).

Many current virtual manipulative environments offer dynamic visuals that attempt to replicate physical manipulatives (Moyer, Salkind, & Bolyard, 2008). As the examples in Figure 1 illustrate, such virtual "replications" are common. In other words, the design of many virtual manipulatives begins with the idea of re-creating the experiences afforded by the physical materials. This transition from the physical world to the digital world has brought with it many advantageous properties including the ability to link iconic and symbolic notations, highlight important aspects, link to other resources, provide unlimited access, and the potential for alteration (Moyer et al., 2002).

Figure 1. Examples of Virtual Manipulatives from the National Library of Virtual Manipulatives

However, the transition from physical to virtual has not been completely beneficial. An obvious limitation of virtual manipulatives is the limited physicality students experience with digital objects. This is due, in part, to the fact that many manipulations in virtual space are controlled using a computer mouse. Despite the ongoing trend shift to more intuitive touchscreen technologies, the vast majority of classroom computers are still controlled by the traditional computer mouse. Manipulating computer mice can be challenging depending on the age of the child and his or her prior experience with such devices. When children are able to control the mouse adequately, there still remains the issue of less physicality overall since moving a mouse is not the same as moving a real world object. How can students differentiate the actions for adding and subtracting objects when both actions are completed with a computer mouse? How does one rotate an on-screen rectangle so that it looks like a diamond? In the physical world, such manipulations are obvious and intuitive to school-age children. But in virtual environments, many of these seemingly minor tasks are less than obvious and can hinder student engagement with underlying concepts.

With this issue in mind, this study conducted a usability test to investigate the challenges young children face when working with virtual manipulatives. Qualitative data, collected through observation, video, and interview, of a second grade student's use of virtual *Base Ten Blocks*, shows that usability issues, originating from input device and interface design, can hinder students' ability to complete designated tasks as well as the likelihood of non-structured exploration with the manipulatives.

Methodology

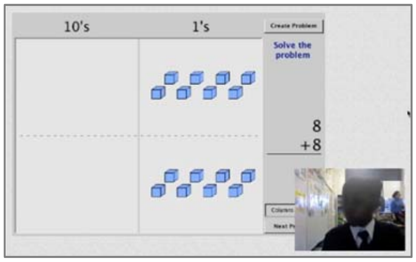
For the study, a seven-year-old male student, named Mike (pseudo name), was invited to participate in the study. Mike was in the second grade. Prior to the usability test, Mike completed a paper-based test composed of 12 addition and subtraction questions using whole numbers from 0 to 12 (e.g., 7 + 8 = ? or 3 + 9 = ?). He answered all of them correctly. After the pre-test, Mike was asked if he uses a computer at home or at school. He reported using a computer at home. He also demonstrated he was comfortable using a computer mouse. At this point, Mike was told that he needed to solve a few more addition questions, but using blocks on the computer this time. For the usability test, a total of five questions were given to Mike. The researcher sat right beside Mike and presented the addition problems that needed to be solved. During the five questions, the researcher asked questions while observing and taking notes on Mike's approach for solving each problem. Further data was collected by recording the computer screen while Mike completed the onscreen activities.. Also, a web camera was used to record Mike's facial expressions during the interview. The entire process lasted approximately ten minutes. The following section describes the findings from these observations and the screen recordings.

Findings

No Manipulation on Virtual Manipulatives

When Mike came to the clinical interview, he was shown two types of manipulatives: physical base ten blocks referred to by the researcher as the "*Yellow Blocks*", and a laptop computer with virtual base ten blocks. The virtual base ten blocks were from the National Library of Virtual Manipulatives (NLVM) and were referred to by the researcher as the "*Blue Blocks.*" Mike was first asked whether he had seen or played with either set of blocks before. He answered, "No." Following this answer, he was asked which set of blocks he wanted to work with. He chose the virtual blocks (e.g., the "*Blue Blocks*"). At this point, the *Yellow Blocks* were put aside, and Mike was instructed to look at the laptop screen. On the screen, there were two columns labeled "1's" and "10's". Each column was divided into two rows. Each row in the 1's column had eight blue blocks, which totaled 16 blocks. Next to the rightmost column was an addition question—8+8—shown vertically. As Mike looked at the screen, he was asked to solve the question. Mike stared at the screen for five seconds, but during that time he did not touch anything. The researcher then suggested that he use the computer mouse. He put his hand on the computer mouse, but again did not move or click the mouse. After another 10 seconds passed, he said, "sixteen." The researcher asked, "How do you know it is sixteen?" Mike answered that he calculated eight plus eight and counted blocks. He seemed to count the blocks using only his eyes to track what was counted.

This might be explained by the fact that he was old enough to solve such a problem in this manner and that the total number of blocks, 16, was not enough for him to need to count by pointing (Geary, Bow-Thomas, & Yao, 1992; Van Luit & Schopman, 2000). However, when Mike had to solve questions using physical manipulatives later, he used a pointing strategy. That is, after he worked with the virtual manipulatives, he was asked to solve the questions 8+3 and 6+7 using the *Yellow Blocks*. Once these questions were given, Mike started to move the physical blocks immediately—arranging them into groups and counting them by pointing with his finger.



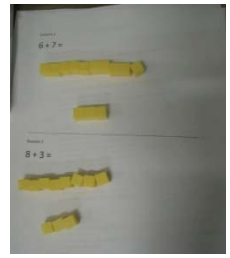


Figure 2. Images of Mike's manipulations of the virtual (left) and the physical (right) manipulatives.

The fact that Mike used the virtual and physical blocks differently suggests that the strategies he used to solve the math questions varied based on the type of manipulative he was asked to use. It might be argued that mental counting with virtual manipulatives is a more advanced procedure than counting with fingers; therefore, asking children to use virtual manipulatives is a more demanding task. However, when we think about *why* physical manipulatives are used in classrooms, it becomes clear that virtual manipulatives may have lost some of their potency as "manipulatives."

Using manipulatives to introduce abstract math concepts has been well supported by distinguished scholars. Pestalozzi (1989), for example, argued that children need to learn through their senses and through physical activity, stating, "things before words, concrete before abstract" (cited in Lutz & Huitt, 2004). Similar to Pestalozzi's idea, Fröbel (1899) developed objects, or "gifts", and emphasized children playing with these objects for cognitive and social development. Montessori (1965) extended the idea of objects and claimed that tactile experience with physical objects allows children to ground their knowledge. Piaget (2001) described ages 7 to 12 as the period of concrete operations and put forth that children in that range demonstrate understanding through logical and systematic manipulation of concrete and related symbols (cited in Lutz & Huitt, 2004).

How well can virtual manipulatives perform as concrete objects that provide grounded experiences for students, when the virtual objects can't be touched or manipulated directly but only controlled by a computer mouse? The fact that many, if not most, digital learning environments require a computer mouse for manipulation, does not seem to support the idea of learning through physical interaction with objects since any interaction with a mouse can be quite challenging, especially for children.

Struggling to Combine Blocks using a Computer Mouse

After Mike's attempt on the first question, 8+8, the researcher showed him how to group the blocks by drawing a square around all of the blocks using a drag-and-drop technique. The researcher demonstrated how this procedure automatically grouped the blocks into one block of ten and six blocks, revealing the answer as sixteen. After a few practice attempts on the first question, Mike was asked two more questions, 5+6 and 9+4. Even after his initial practice, it took approximately two minutes for mike to solve 5+6 and one and a half minutes to solve 9+4 by grouping the *Blue Blocks* using the drag and drop technique. For the 9+4 question, the researcher had to demonstrate, again, how to draw a big square to group all of the blocks on the screen. When he was solving the question 9+4, it was observed that Mike was still struggling to draw a big square around the base ten blocks. After several unsuccessful trials, the researcher asked him, "Mike, what are you trying to do now?" Mike answered, "I'm trying to make ten." That is, instead of drawing a big square around all 13 blocks on the screen, he was trying to draw a square around a group of 10 blocks. This suggests that he knew how to solve 9+4 by grouping, and was trying to do a more advanced strategy, which was grouping 10 blocks first and then counting the remaining blocks. Ironically, the software did not allow him to use this more advanced strategy. The software would only accept his input if he drew a square around *all* the blocks on the screen (in this case 13 blocks)—only then would it visualize a group of ten and three blocks.

After completing three questions with the virtual manipulatives (*Blue Blocks)*, Mike's laptop computer was taken away, and he was given the physical blocks (*Yellow Blocks)* and a sheet of paper with two questions: 8+3 and 6+7. He was asked to solve the questions using the *Yellow Blocks*. As soon as the instructions were given, he started to put *Yellow Blocks* together and he solved the question in 20 seconds. He solved the next question in 10 seconds. Lastly, he was asked by the researcher, "There are two more questions that you need to solve. Which blocks do you want to use to solve those questions?" He took a couple of seconds and answered, "the *Blue Blocks*." The researcher then asked him a follow-up question, "Why do you want to use the *Blue Blocks* instead of the *Yellow Blocks*?" He replied, "Because it is easier to use a computer."

Discussion

One noticeable observation was that Mike initially did not touch either the computer mouse itself or the on-screen virtual objects when counting. Instead, he counted the objects with his eyes even after he was instructed to use a computer mouse.

Another observation was that Mike struggled to use the click-drag-drop motion required by the computer mouse. He spent almost two minutes trying to draw a big enough square to cover 11 blocks. This observation raises some questions such as how intuitive are common mouse actions such as clicking, dragging and dropping to young children? It's not a new idea that young children have more difficulty using a computer mouse compared to adults (see Donker & Reitsma, 2007a, 2007b; Joiner, Messer, Light, & Littleton, 1998). An important question that needs to be addressed is how meaningful are those motions and/or gestures in children's understanding of mathematical concepts?

Gestures, movements, and other physical activities have been considered crucial factors in cognitive processing, and research has shown that the role of body and gesture is a significant factor contributing to student understanding in mathematics learning (Alibali & Nathan, 2012; Goldin-Meadow, Cook, &

Mitchell, 2009). For example, Lakoff and Núñez (2001) state that human motor-control systems may be centrally involved in mathematical thought. They further argue that the word *add* has the physical meaning of physically placing substances or a number of objects into a container and explain that addition can be understood as "putting collections together" (p. 55). Based on these ideas, it is suggested that activities with physical manipulatives, such as *putting blocks together,* can be a key component of fostering understanding of concepts such as addition.

Unfortunately, since many physical manipulatives have become virtual objects, physical actions such as "putting" have been removed or distorted. Instead, students take actions indirectly, by clicking, dragging, or dropping with a cursor and a computer mouse. For example, to put blocks together in the base ten manipulative environment studied here, students draw a square around blocks using a drag and drop motion. To turn shapes, students click a button with an arrow icon. To add blocks, students click a button with a "plus" sign and to take away blocks, they click a button with a "minus" sign. Of course, text, graphics and images keep changing visually and visualization is an important part of young children's mathematical development. However, it is unlikely for young children to associate the visual representation of mathematics concepts with their actions, when the manipulations are only limited to possible actions performed with a computer mouse. Wouldn't it be more beneficial for young children to actually turn a square 90 degrees to learn the concept of a diamond as opposed to clicking on a button?

The last observation worth of highlighting is that despite struggling for two minutes to solve one question, Mike wanted to continue working the virtual manipulatives rather than the physical manipulatives. Based on his response, it seems he believed that it was easier to use the computer than the physical manipulatives. Obviously, many young children prefer to work on computers than to work with physical blocks since for many children using computers at school is still a bit of a novelty. What needs to be highlighted though is that Mike did not stop trying to solve a question after struggling for two minutes. It is not known what Mike was thinking while he was trying to solve the question using a computer mouse, but it is clear that he was cognitively active while he worked behaviorally to manage the mouse. This anecdote illustrates the potential of virtual manipulatives to engage young children in mathematics activities despite the struggle with manipulations.

Conclusion

The observation and interview with Mike highlights the limitations of virtual manipulatives as well as their potential to be used as instructional tools for mathematics learning. While the study shows that it was challenging for him to manipulate the virtual objects using a computer mouse—the required movements were not intuitive to Mike, but it was also encouraging to observe how much effort he put into making the virtual manipulatives work. Also, considering the challenges that Mike encountered were from limited interaction due to using a computer mouse, it is exciting to think about the potential of new input devices such as a touchscreens and gesture detection to overcome these challenges.

Thus, our tasks as researchers, instructional technology developers, and educators have become slightly clearer. Researchers need to continue to investigate how virtual manipulatives can be designed to be effective instructional tools. Developers should continuously adopt and apply research findings to design and develop instructional tools while they look for the proper technologies to support teaching and learning. Lastly, educators should continue to search for developmentally and cognitively appropriate tools for young children, but also to seek the best pedagogical practices to implement such tools into the classroom setting.

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RE-CONCEPTUALIZING PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN CALCULUS

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What do students need to know and understand to succeed in college calculus? Three new research dissertations investigate the knowledge level of students concerning derivatives, continuity and series convergence. Findings of these studies identify student success and difficulties with the concepts of notation, terminology, functions, infinity, limit, continuity, images of change, multiple representation, composition and decomposition. Weak procedural knowledge does not transfer to other problem representations or topics. Strong students are concerned with conceptual understanding, making connections to previous concepts; they are divergent thinkers. The curriculum, the instructor, the student's peers, and oneself can initiate such a re-conceptualizing approach.

Over 600,000 students take introductory calculus every year in U.S. (Bressoud, 2004). Calculus is a first step in fulfilling: (a) the future need for more scientists, technicians, engineers, and mathematicians (the STEM supply pipeline); and (b) the necessity for more innovative workers (a knowledgeable population) trained in science, technology, engineering and mathematics (Zollman, 2012). However, Peterson (1986) says that half of the students taking an introductory college calculus course fail every year. And according to Cipra (1988) and Peterson (1986), even students who pass introductory calculus still have poor calculus skills. Selden, Mason, and Selden (1989) state that calculus students do not understand the fundamental concepts. These students are unable to solve non-routine problems. Their research suggests that calculus students recall very little calculus in later classes (Selden, Selden, Hauk & Mason, 1999; Selden, Selden, & Mason, 1994).

What do students need to know and understand to succeed in college calculus? Why do even our best students not retain knowledge? Is it just a question of conceptual knowledge not being attained, or is there more for us to understand in the learning of calculus? These are parts of research questions of three new dissertations investigating the knowledge level of students concerning derivatives, continuity and series convergence (McCombs, 2013; Patel, 2013; Wangle, 2013). These dissertations were selected meeting three criteria: (a) theses dissertations investigate three major troublesome calculus topics of students; (b) these dissertations used an APOS theory framework (Dubinsky, 1991) to study student understandings; and (c) these dissertations were completed within the past 6 months and have not been published in journal articles.

Student Understanding of Derivatives

Patel (2013) investigated college student understanding of derivatives and their connections to function, slope, rates of change and limits. Her mixed methods study was conducted in six beginning calculus classes over 15 weeks that included quantitative content assessments and disposition surveys followed by classroom observations and a teaching experiment with student interviews. Findings indicate that by the fourth week of the semester most student responses are at an Action level with derivatives but few student responses develop to a Post-Action level at any time during the semester. APOS theory is framework for research and curriculum development in undergraduate mathematics education, based upon Dubinsky's elaboration of Piaget's notion of reflective abstraction. The core of the framework is the theoretical perspective that all mathematical conceptions can be understood as actions, processes, object, or schemas – thus APOS (Breidenbach, et al., 1992; Dubinsky, 1991; Dubinsky, et al., 2005a; 2005b).

Student responses only at the Action level restrict student use of derivatives or their connections of derivatives to functions, slope, rates of change, limits, and applications. Using the findings of her teaching experiment and interviews, Patel suggests that students need a situated learning environment that allows students to become reflective learners. She also recommends that the curricula integrate multiple representations of slope, rates of change, limits and derivatives.

Student Understanding of Continuity

Wangle (2013) researched college students understanding of continuity in beginning calculus. She did a mixed methods approach, using quantitative findings from function, limit, and continuity instruments to identify high-ability, average-ability, and low-ability students for in-depth interviews. As others researchers found (Bezuidenhout, 2001; Tall, 1990; Vinner, 1983; Williams, 1991) Wangle observed that students confuse continuity with the function being defined, the limit existing, differentiability of the function, and graphical connectedness (don't raise the pencil test). As Castillo-Garsow (2012) describes, Wangle noticed most calculus students only think of functions as chunky, not smooth, when reflecting on change. Low-ability students had significant difficulty finding the domain of a function. Average-ability students were inconsistent in their methods of solving problems and very dependent on graphical representation of a function. High-ability students had a large example space, were able to reason with properties of function, recognized the necessity of reasoning consistently with the algebraic and graphical forms of the same function, and tended to be "smooth" thinkers. Wangle advises the use both algebraic and graphical forms of function and emphasize strengths and weakness of each in the context of the problem being solved. She advocates a variety of functions when showing students examples to help students develop their own examples to make sense of concepts, and to challenge potential contradictory concept images. She also suggests an emphasis on comparing and contrasting the interrelationship between limit, continuity and differentiability.

Student Understanding of Series Convergence

McCombs (2013) studied more than 40 second-semester calculus students' understanding of series and series convergence. He did a mixed methods approach, using quantitative findings from function, infinity, limit, and sequence instruments to identify high-ability, average-ability, and low-ability students for in-depth interviews. Regression analysis found no correlations between series convergence and function instrument or infinity instrument. Correlations were found between series convergence and limit instrument and sequence instrument. Students confused sequence concepts with series concepts; did not know which series test to use; did not know conditions or conclusions of series tests and thus did not use the various tests correctly. Students' limited algebra and arithmetic abilities did not allow the students to work tasks completely. In the interviews, McCombs observed students' correct written work masked what they knew or understood concerning series and series convergence. Students commented their solutions methods were based upon the examples they had seen before, either in class or textbook. All of the low-ability and most of the average-ability student solutions were at the Pre-Action APOS level. Small variations in the notation or format of exercises changed a student's response APOS classification. High-ability students were able to verbalize their procedures and to relate prior knowledge of similar tasks previously given, to understand a new task. McCombs concludes that the curriculum needs a variety of APOS-level tasks. Series convergence tests need to be presented in multiple representations (numerical, graphical, algebraic) to aid students to develop beyond the Action level. He proposes instruction with multiple representations and multiple reflective abstraction initiates – including common misconceptions for cognitive dissonance – as suggested by Cappetta & Zollman (2009; 2013).

Summary

As other researchers have found (Cappetta, 2007; Castillo-Garsow, 2012; Cipra, 1988; Peterson, 1986; Tall, 1990; Vinner, 1983), students beginning at the university in calculus class have a fairly strong procedural knowledge and a fairly weak conceptual understanding of mathematics. Most students use only their strengths, not their weaknesses in calculus class. They keep a mindset of mathematics that hinders their deeper learning of calculus. High-ability students in calculus, on the other hand, have openness to working with mathematics. While these students also begin with their strengths, they venture into less comfortable areas while studying and reflecting on the mathematics. These students think of relationships and functions in terms of both smooth and chucky rates of change. High-ability students have a repertoire of representations they used depending on the context of the mathematics. They seamlessly can move between numerical, algebraic, graphical and application representations. High-ability students view composing and de-composing processes as a means for comprehending mathematical procedures, definitions and theorems. High-ability students are concerned with conceptual understanding so they look to make connections to previous concepts; they are divergent thinkers with the ability and persistence to seek multiple representations of mathematics (Zollman, in press). It appears what is needed is a situated learning environment for fostering the student to become a reflective learner. The curriculum, the instructor, the student's peers, and oneself can initiate such an approach to re-conceptualize procedural and conceptual knowledge (Cappetta & Zollman, 2009; 2013). Student responses need to be at an APOS Process level for retention and transfer of knowledge.

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EXTENDING MATHEMATICAL DISCOURSE

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This preliminary study explored the use of SmartPen technology with pre-service elementary teachers who are reasoning about rational number tasks in their methods classrooms. It discusses how Pencasts, created using the SmartPen and shared between classrooms, impacted student discourse about and understanding of rational numbers. The portability of Pencasts between classes allowed us to transport divergent thinking between classes, and have students consider ideas that may not have otherwise emerged within a particular class. The effect was to create a more diverse and rich discourse.

One of the perennial issues in fostering discourse in mathematics classrooms is getting students to listen to and understand each other's arguments. Moreover, students are often reluctant to respond to each other's argument in a critical way. A second and related problem occurs as teachers move around the room anticipating how they will structure discourse. Often, student explanations that are desirable for the ensuing discourse disappear as they converge with that of tablemates or overheard from adjacent tables. We wondered if and how technology might affect this process. We found that, not only were we able to have students write and react to one another's thoughts within classes, we were able to preserve and share authentic student thinking between classes. This allowed us to broaden discourse and encouraged our students to analyze, understand, and critique alternative explanations; thus deepening their own understanding.

In this paper, we describe a teaching experiment involving the use of what we will call a SmartPen in our mathematics methods classes for preservice teachers. After explaining what we mean by a SmartPen, we will describe a teaching experiment involving the use of this device with our pre-service teachers as they explored rational-number tasks typical of the middle-grades curriculum. Finally, we will consider pedagogical aspects of using the SmartPen; speculate on how its use might facilitate or enhance class discourse and possible other benefits or concerns about using this technology to promote mathematical reasoning.

SmartPens

A SmartPen is essentially a thick stylus with both a ballpoint and a scanning device in the tip. It is used to write on special paper containing microdots. The microdot paper may be printed, as needed, on a laser printer or purchased in various notebook formats from the manufacturer.

When writing on the paper (recording a Pencast), miniature audio components in the SmartPen allow any spoken narration or commentary to be recorded synchronously as the pen scans the written work. When the SmartPen is connected to a computer, the recorded session can then be uploaded, edited, and saved in several formats; including a PDF file with embedded audio. When the Pencast is replayed, the writing appears as it occurred along with any dialogue that transpired. The main advantage the SmartPen afforded was the simplicity of production. With minimal instruction, students could write on the paper in their own, natural style to produce a Pencast; either at the document camera or at their own tables.

Theoretical Framework

Numerous studies have noted the importance of communication in the teaching and learning of mathematics (Cobb, Wood, & Yackel, 1991 and Lampert, 1990, among others). Others have observed that discourse fosters children's mathematical thinking (Davis, 1997; Kazemi, 1998; Lo & Wheatley, 1994). NCTM has consistently advocated the importance of discourse. Standard 5 of the standards for the teaching of mathematics holds teachers responsible to orchestrate discourse by posing questions that elicit, engage, and challenge students' thinking; by listening carefully to students' ideas; encouraging and accepting multiple representations, and by asking students to clarify and justify their ideas (Martin, 2007). White (2003) notes that properly managed discourse allows students to concentrate on sense making and reasoning. Moreover, discourse allows teachers to reflect on students' understanding and plan to purposefully stimulate their mathematical thinking. We adopt Ball's (1991) perspective that discourse is mutually formed at the confluence of students, teachers, and the tools they use. We considered Pencasts to be a possible tool to further discourse.

We theorized that the use of Pencasts might support classroom discourse in several possible ways. First by providing a means of introducing alternate, authentic student thinking as the class considers a mathematical idea. Secondly, the SmartPen technology might enable the preservation of useful preliminary notions or alternative thinking. Finally, it may provide a means of having a number of readily accessible explanations to bring into our preservice teachers' consideration. The research question that emerged from these ideas was; Does the use of Pencasts affect discourse and thereby student mathematical thinking? If so, how and in what ways?

Methodology

This preliminary study used qualitative, action research methods (Hubbard & Power, 1993, Stringer, 1999) to address the questions and analyze the data. At the time we decided to try the use of the SmartPen with our students, we were each teaching separate sections of a mathematics methods class for preservice elementary teachers working towards a K-8 Certificate (N=37). The participants were predominantly white, female, and traditional post-high school students. Other ethnicities, males, and non-traditional students comprised less than 10 percent of each class, respectively.

We purposefully selected students to make pencasts as we moved about each class listening to their explanations. We intentionally sought divergent, plausible thinking and progressions of thinking that could potentially support ensuing discourse regarding mathematical ideas. In this way, we functioned no differently, from a teaching perspective, in how we normally chose to set up discussions of the problems we considered. The data for this report consist of the pencasts that students created and field notes of the ensuing class discussions, both those involving the viewing of Pencasts and those without. The data were examined for emerging themes to be analyzed for possible explanations and determine potential actions to be implemented in the classrooms.

Both classes were involved in problem solving with rational numbers using similar tasks. The tasks were chosen to be consonant with the recommendation of the Common Core State Standards for Mathematics (CCSSM). For example, the CCSSM Grade 7 Standards for Mathematical Content call for students to "solve real-world and mathematical problems involving the four operations with rational numbers" (p. 49), and also to "Analyze proportional relationships and use them to solve real-world and mathematical problems" (p. 48). Also, we were mindful of the CCSSM's Standards for Mathematical Practice (SMPs). For example, students should not only be able to "justify their conclusions, communicate them to others," but also to listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments." (SMP 3, pp. 6-7).

Results

Classroom 1 was the first to produce a Pencast session. This class had established a norm of inquiry. Students were encouraged to understand the *why* behind the *what* of the mathematical ideas they encountered. They were used to sharing their thinking with each other. One problem they worked on was. "*Ming ran* $1\,{}^{3}/_{4}$ *km and then walked* ${}^{5}/_{6}$ *km. What was the total distance that she covered?"* Hannah's Pencast described her thinking, and her classmates commented on her ideas. Recording the Pencast at the document camera was the only departure from the normal class routine of displaying their already written work and "briefing" their thinking to the class. Classroom 2 had also been looking at similar problems. Hannah's Pencast was introduced into their discussion. They were surprised by Hannah's approach of looking at the distances not traveled since all of Classroom 2's students took an *additive* approach to the task by building up the actual distances traveled ($1\,{}^{3}/_{4}$ km + ${}^{5}/_{6}$ km).

Figure 1 presents only a static picture of Hannah's written record akin to a notebook record that would be, "briefed." As such, represents a static, limited view of the thinking. The entire Pencast (including the audio) can be viewed at the following link: http://rathsongard.net/pencasts.html.

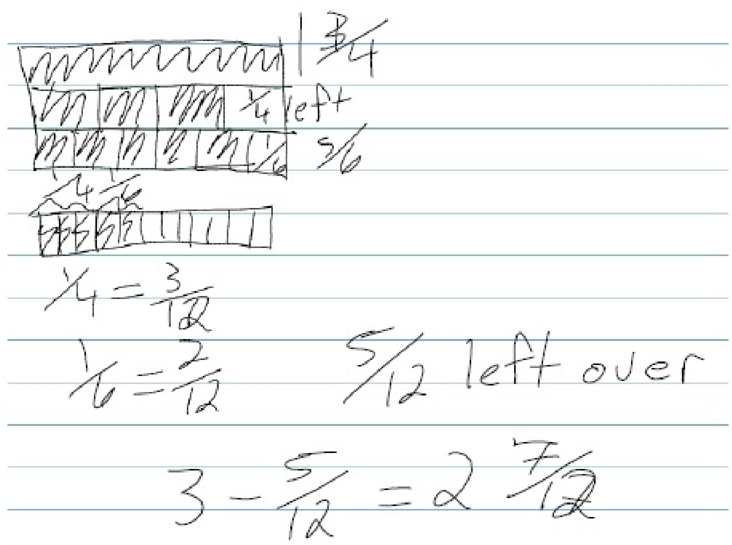

Figure 1. Hannah's Reasoning

The dynamic aspect of "seeing" and "hearing" another student's reasoning was significant for Classroom 2. A number of mathematical ideas surfaced. Students noted how Hannah describes having "…three [km] is the whole for me." Then she describes the use of common denominators to determine the amount of the three kilometers *not traveled*: Namely, she describes "${}^{1}/_{4}$ [km] left here" (meaning a distance *not ran*) and "${}^{1}/_{6}$ [km] left here" (meaning a distance *not walked*). Thus, Hannah used converting and combining the unit fractions (${}^{1}/_{4}$) and (

$^{1}/_{6}$) to get $^{5}/_{12}$ km *not traveled.* Hannah is also heard describing how the distance actually traveled would be (3 - $^{5}/_{12}$) = 2 $^{7}/_{12}$ km.

The ensuing discussion considered how middle-school students might approach the problem if encouraged to use their own thinking to problem-solve. Students in Classroom 2 readily admitted that “rote training” played a large part of their automatic reach for an additive strategy. In discussing their own approaches to the problem, a number students commented along the lines that “It’s an addition problem, right?” or that “You’re supposed to add.” Reacting to Hannah’s work, one student said that “It never occurred to me to do anything other than add the numbers…” This led to a discussion about relevance or appropriateness of the terms “right” and “wrong” in mathematics teaching and when it came to considering divergent ways of problem-solving.

The value of replaying and reflecting back on Pencast sessions became apparent over time and was reinforced when both classes shifted from additive thinking with rational numbers into problems involving multiplicative thinking. One problem, read as follows: “*For a worker’s daily ration, one full portion of bread is two-thirds of a loaf of bread. How many portions can be found in 1 $^{4}/_{5}$ loaves?”* Again, we had our students use the SmartPen to present what they thought. One student, Jessica, gave a seemingly convincing argument for the answer being “2 $^{7}/_{15}$” because she had a compelling narrative to accompany her visuals (See Figure 2 and http://rathsongard.net/pencasts.html). Both classes considered her explanation.

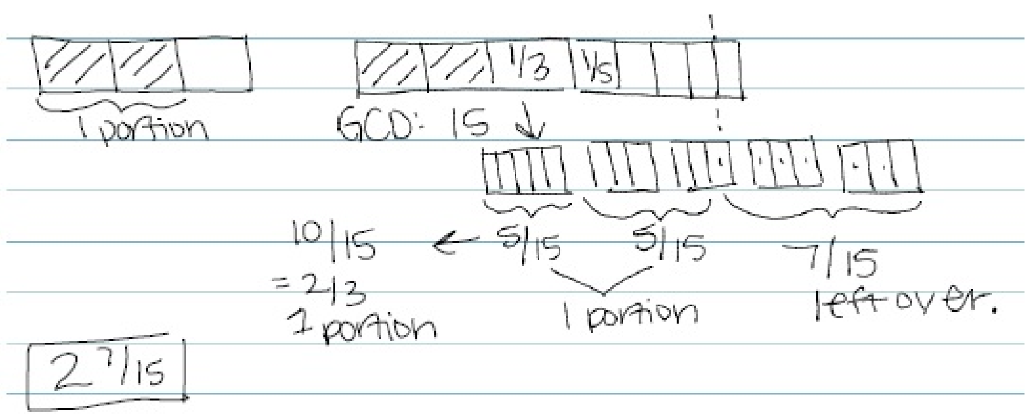

Figure 2. Jessica’s Explanation

After viewing her Pencast, many students in both classes questioned their own (correct) answers of “2 $^{7}/_{10}$”. Jessica first identifies and labels “1 portion” as $^{2}/_{3}$ of a loaf. She then shows the total amount of bread by having one loaf broken into thirds and a second loaf broken into fifths (indicating how she only had four-fifths of the latter loaf). After easily taking “1 portion” from the first loaf, Jessica focused on the remaining bread (the $^{1}/_{3}$ loaf and the $^{4}/_{5}$ loaf). Approximately 3:50 into the Pencast, Jessica describes her sense that “…of the $^{1}/_{3}$ and the $^{1}/_{5}$, I needed to find a common denominator, so it’s 15…”, and then finds another “1 portion” by locating $^{10}/_{15}$ of a loaf which she justified as equal to $^{2}/_{3}$ of a loaf. Having accounted for what she had apportioned, she noted “…and then what I had left over was my $^{7}/_{15}$.” After, watching and listening as Jessica presented her thinking, the students agreed that the explanation makes a plausible case for an answer of “2 $^{7}/_{15}$”.

Amanda’s Pencast was comparable in her initial approach to the loaves and portions. There is a key part in her Pencast Session where she clearly has found 2 portions in her 1 $^{4}/_{5}$ loaves. However, the remaining “$^{7}/_{15}$” again figures prominently in her thinking (Figure 3 and http://rathsongard.net/pencasts.html). At about 2:30 in the Pencast, Amanda can be heard to reflect on what the problem in fact asked. She refers to the constitution of “1 portion” and the “$^{7}/_{15}$” remaining. Noting that, “I had to compare it to the portion size.” After re-drawing the “1 portion” as two-thirds of a loaf, she observes:

> “Within $^{2}/_{3}$ there’s $^{10}/_{15}$, so I knew that I had to base it upon 10 instead of 15, and there were 7 in there…if that makes any sense [laughs]. I don’t know what I just did. So yah I had 7 leftover, so 7 of 10, so it would be 2 $^{7}/_{10}$.”

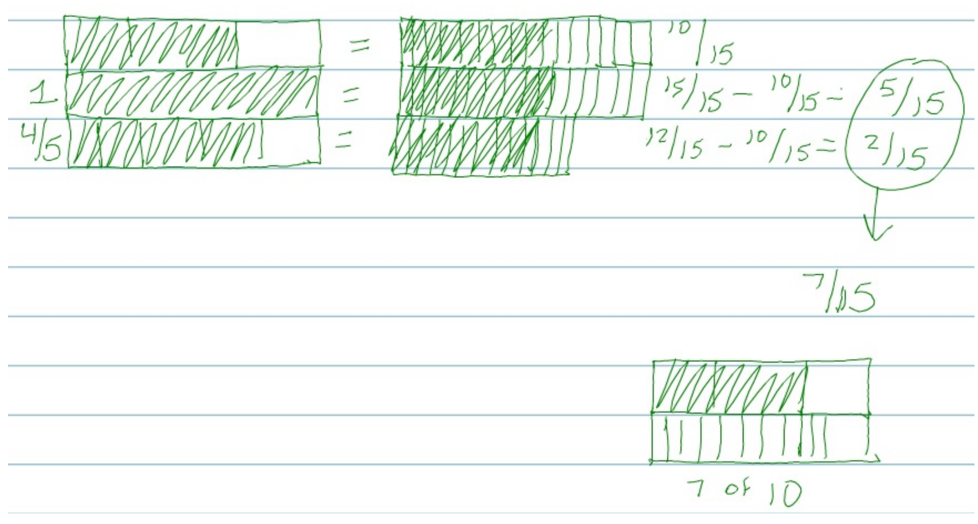

Figure 3. Amanda’s Thinking

Asked to re-explain how she got the “$^{7}/_{10}$”, she first offers disclaimers, “I don’t know how to explain it.” and “It’s kind of confusing.” Then she goes on to appropriately describe “1 portion” as having 10 *parts* and how she, “…only used 7 of the 10.”

Over subsequent discussions in both classes, it became clear that it was not enough to "agree to disagree" on different answers ($2\ ^{7}/_{15}$ vs. $2\ ^{7}/_{10}$). The preservice teachers wanted to understand exactly why two arguments could be so convincing (as in Jessica's Pencast) yet result in different answers. As they discussed the explanations, they decided that careful attention to the unit being counted was an important factor in distinguishing mathematically appropriate explanations.

Discussion

The portability of the Pencasts between classes allowed us to introduce and consider thinking that may not have otherwise emerged within a class. This enabled us to introduce authentic student thinking which had the effect of broadening discourse and encouraging students to analyze and critique alternative, plausible explanations. This may improve a teacher's ability to make consistent, intentional, and strategic choices about the thinking surfaced in their classes. If desirable ideas do not emerge, additional authentic student voices can be brought to bear. Another interesting tendency regarding discourse emerged. Considering reasoning provided by an unknown student could possibly create a sort of "emotional distance" that may allow students to focus on and respond to the reasoning offered rather than any personal relationship with the source of the reasoning; thus further opening up discourse.

It is valuable to have an actual record of what was written and said, what was discussed in class, and to be able to call on or refer back to this record as learning progresses. Not all Pencasts generated a lively discussion or reaction. By capturing *everything* that was written and spoken, they afforded opportunities for both teachers and students to deliberately choose which sessions were reviewed. Pencasts appeared to enable more diverse and rich discussions. A related consideration is that teachers sometimes have to introduce alternate explanations if they do not emerge in the class. Students have a tendency to interpret this as conferring authority to the perspective. This did not appear to be the case with Pencasts. They seemed to accept, regard, and respond to Pencasts as if they had been generated by students within the class; without conferring authoritative status.

The introduction of Pencasts seemed profound for our preservice teachers. On the one hand, they are learning to think as classroom practitioners and diagnosticians. On the other hand, they are also learning to think of the actual mathematics at a greater depth. Rather than rely on rote computation (as in the division-of-fractions problem), they developed more facility at not only organizing and explaining their own thinking but also understanding and examining that of others and recognizing the importance of careful attention to the unit (SMPs 2, 3 & 6). Distinguishing appropriate reasoning, when considering seemingly equally plausible explanations, became a key component of furthering their understanding of the mathematical relationships and the importance of discourse. They realized that middle-grades learners were apt to demonstrate a similar range of thinking. Our preservice teachers appreciated being able to point to each other's work for examples of the kinds of thinking that might be expected from their own students.

We found the SmartPen to be precise and students were able to naturally express themselves while avoiding the learning curve that often accompanies the introduction of new devices. This preliminary study has provided several possible avenues for future research. The various discourse effects that emerged merit examination in more depth. Another possibility is to examine the use of other technologies such as tablet applications with comparable features to determine if they might have similar effects.

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TASK ALIGNMENT TO THE COMMON CORE: HOW OUR SOLUTION LENS MATTERS

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This paper reports on exploratory research examining the relationship between choice of solution strategy when solving a task and perceptions of content of the task. Data were gathered from 56 teacher-participants who examined a task containing numerous connections to common core standards for mathematical content. Participants predicted the solution and explained their rationale before solving the task and identifying content standards in the task. Explanations for predictions and solutions were coded and used to compare and contrast the standards identified with strategies used to solve the task, thus providing information how strategies likely mirror perceptions of the connection to standards.

As teachers transition to teaching mathematics aligned with content of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), it is important for them to be aware of connections within the mathematics, and view the content cohesively. As tasks are developed for new materials to address the content of the CCSSM, teachers must also consider the mathematical content addressed in these tasks. A carefully designed mathematical task may elicit several viable problem-solving approaches from which different content emerges.

Our theoretical basis is situated within the work on task design discussed in the International Commission on Mathematical Instruction (ICMI) Study 22 (Margolinas, 2013a). In describing the rationale for the document Margolinas (2013b) notes:

> There has been a recent increase in interest in task design as a focus for research and development in mathematics education. This is well illustrated by the success of theoretically based long term design research projects in which design and research over time have combined to develop materials and approaches that have appealed to teachers. One area of investigation is how published tasks are appropriated by teachers for complex purposes and influence mathematics teaching. (para. 1)

Arguably, in order to understand such appropriations and influence on teaching, we must first understand how teachers, themselves, engage with the tasks and align the mathematics of the tasks with standards even before appropriations for teaching purposes occur. Chevallard (1989) describes the ternary relation between the teacher, the student, and the knowledge to be taught as a didactic relation. In conceptualizing this notion, Chevallard identifies a difference between *used knowledge* and *taught knowledge*, where the former refers to knowledge used by a person to do something and the latter refers to knowledge that is not free from context. In the case of mathematics education we consider the following: How might teacher's *used* mathematical knowledge relate with the *taught* mathematical knowledge they seek for students to develop?

The data reported in this paper address the following research question: How do prospective teachers solve a mathematical task appropriate for the grade level they teach, and what CCSSM content do they identify as related to the task? We are particularly interested in understanding if the mathematics content connections pre-service teachers make between the task and the standards document are limited to the mathematics they use in their solution strategies. To examine this, our research involved a task with multiple mathematical content connections and a variety of entry points for solution strategies connected to CCSSM standards in geometry, measurement, or ratio and proportions with regard to properties of scaling.

Methods

Data were collected from 56 participants enrolled in one of four college-level mathematics education classes at one of two institutions of higher education. The participants were predominantly working toward teacher certification, whether in a baccalaureate pre-service program or a post-baccalaureate program as follows: BEd in mathematics, 24; Post-BEd preparation certification in mathematics, 7; MEd in mathematics, 4; and other (doctoral student or non-mathematics program) 7. Fourteen participants did not provide status information.

The task was piloted in Spring 2013 resulting in modifications that yielded the final set of prompts used to gather data during Fall 2013. The task asked participants to examine three figures and make a prediction, solve the problem, and provide rationales for both prediction and solution. The task, as pictured in Figure 1, involves 3 figures of circles within squares that are configured in a specific manner to explore which of the figures has the most area outside the circle(s) and inside the square. We were most interested in participants' responses to the prediction and rationale requested in Prompts 1 and 2 prior to solving the problem and the validation requested in Prompt 4. These data would provide information on these prospective teachers' thinking about the problem and how they approached solving it.

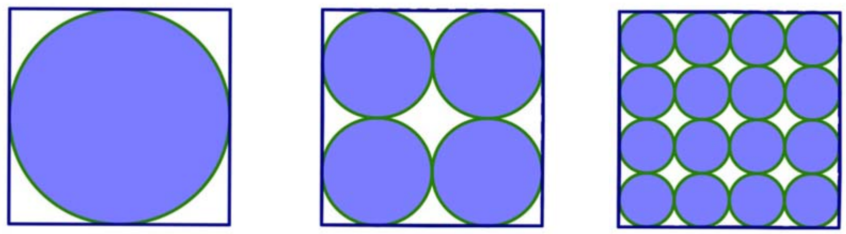

Figure 1 Figure 2 Figure 3

Participants completed the prompts related to this problem.

1. Assuming the three squares are congruent, chose 1 prediction.

   Figure 1 has the most area not shaded blue

   Figure 2 has the most area not shaded blue

   Figure 3 has the most area not shaded blue

   All three figures have the same area not shaded blue

2. The rationale behind my prediction is as follows:

3. After finding a solution, the following choice is my answer. (Participants had the same choices as in Prompt 1).

4. Provide a description of your solution strategy.

5. Examine the Common Core State Standards for Mathematics. Identify specifically to which standards you feel this problem is well-aligned.

Figure 1. Circles and squares task.

Participants' choices for Prompts 1 and 3 were tallied. Explanations for predictions and solutions (Prompts 2 and 4 respectively) were coded for 1) the basis for thinking about or approaching the problem and 2) the processes used in predictions and solutions. The basis-for-thinking codes represent participants' conceptual or algorithmic frameworks and the process codes describe the mental actions taken. Two of the authors analyzed the written responses for Prompts 2 and 4 looking for emerging themes. We used constant comparative analysis beginning with an initial data analysis by class group and continuously comparing and refining categories with analysis of subsequent class groups. As new data are constantly compared with previous data, relationships and patterns may be discovered (Goetz & LeCompte, 1981). Examples to clarify the codes are provided in Tables 1 and 2.

Basis-for-thinking codes. Table 1 provides descriptions of the basis-for-thinking codes with illustrative examples from either student work on the prediction or the solution.

Table 1

Basis-for-thinking codes and illustrative examples

1. Ratio/proportion. Uses the similarity of circles and squares (also congruent) to imply that the ratios of shaded to non-shaded areas in the three diagrams will be equal.

    "Figure 2 can be represented as 4 images of figure 1 with 1/4 the sides of the square and radius. The ratio of shaded non-shaded does not change since the squares are congruent." (Prediction)

    "The squares are congruent and the circles in figure 2 and 3 are similar to the circle in figure 1. Therefore the area[s] of the circles are the same in each figure." (Solution)

2. Equivalence. Uses equivalent expressions to represent the total area of all the circles in the three diagrams. Some note the inverse relationship between the number of circles and the area of each circle.

    "The circles are one-four-sixteen. All fitting into the SAME area. If R = 4, 4x4x3 ~ 48; If R = 2, 2x2x3x4 ~ 48; If R = 1, 1x1x3x16 = 48" (Prediction)

    "*x* = # of circles enclosed figure 1: *X*, figure 2: (1/4)**X**4= *X*, figure 3: (1/16)**X**16=*X* All circles are the same amount of shading. (Solution)

3. Pattern/rule. Bases response on the regularity in the number of circles in each of the three squares.

    " 'All three figures have the same area not shaded blue.' The reason being is that I saw a pattern with the rule being 'multiply 4'. (Prediction)

4. Formula/calculation. Uses formula or calculation with specific assigned values, generalized quantities, or actual measurements from the given figures.

    "I plugged in some numbers for the area of the square, as well as used these numbers to help me find the radius of the circle to find its area." (Prediction)

    "Set up the side of the square as *X* and calculated the area of not-shaded in each figure. The areas of not-shaded are equal." (Solution)

5. Rearranging. Visually rearranges the white areas within each diagram to assess the total white area for that diagram.

    "…if I were to manipulate the unshaded parts, it would be a greater amount with figure 3." (Prediction)

6. Limit. Uses calculus principles related to area under a curve or filling a three-dimensional space.

    "My reasoning is similar to the reason that when trying to find the area under a curve smaller segments are more accurate…." (Prediction)

7. Guess/estimate. Respondent states that she/he is guessing or estimating.
8. Model. Uses a model (a diagram) in solving the problem.
"I drew a model of the problem and assigned a random number for one side of the square…." (Solution)

Basis-for-process codes. Table 2 provides a description of the basis-for-process codes with illustrative examples from either student work on the prediction or the solution.

Table 2

Basis-for-process codes and illustrative examples

1. Reasoning. Uses relationships, sometimes noting proportionality, to come to conclusions about the solution.
"Since the square is the same size in each figure, it gives the circles inscribed special qualities that keep the areas the same." (Prediction)
"No matter how many circles are in the square, the length of the side of the square will not change. Therefore the radii add up to the first square." (Solution)
2. Comparison. Finds the difference between the area of the square and the total areas of the circles.
"I thought about the math I would need to find the area that is not shaded by taking the difference of the area of the squares and the circles." (Prediction)
"…I then subtracted the area of the circles from the area of the square." (Solution)
3. Visual observation. Makes a visual assessment.
"The sum of the diameters cover[s] the length of the square about the same from looking at it." (Prediction)
4. Over-generalization. Mistakenly applies a generalization from a different context to this problem.
"In biology something we learn is that the best way to increase volume in a small area is to increase surface area…." (Prediction)

For Prompt 5, standards participants identified as connected to the question were tabulated.

Results and Discussion

We share participants' predictions and solutions to the task, coding for thinking and processes, standards they viewed as well-aligned, and compare and contrast standards identified related to the task with the strategies used to solve the task. Thus, the data provide insight on the solution strategies and perceptions of the connection of the task to the CCSSM.

Table 3 provides data relative to choices made for the predictions of the participants, with about two-thirds selecting the correct solution.

Table 3

Number Of Participants Who Predicted Each Solution (N = 56)

Choice	Number
Figure 1 has the most area not shaded blue	8
Figure 2 has the most area not shaded blue	1
Figure 3 has the most area not shaded blue	9
All three figures have the same area not shaded blue	38

Even for some of these participants, who by and large would have studied the mathematics related to the problem, predicting the solution was not obvious. That is, given a problem situation in which ideas studied must be applied, there is something to consider with regard to *predicting*. However, when determining solutions, 54 of the participants stated that all three figures have the same area not shaded blue, while one chose Figure 1 and one chose Figure 2.

Thinking Codes and Basis-for-Thinking

The coding frequencies are given in Table 4, showing more variety in the thinking about the predictions than when determining the solutions.

Table 4

Codes and Frequencies for Basis-for-Thinking on Prediction and Solution

Code Number	Code Description	Frequency on Prediction	Frequency on Solution
0	No Response	3	4
1	Ratio/Proportion	14	2
2	Equivalence	8	9
3	Pattern/Rule Estimation	3	0
4	Formula/ Calculation	7	44
5	Rearranging	4	1
6	Limit	7	0
7	Guess/Estimate	11	0
8	Model	0	1
9	Other	3	2

Note. Some responses were coded into two categories. Hence, column total is greater than 56.

Process Codes

The coding frequencies for the processes used on prediction and solution are given in Table 5. While there were a large number of no responses in both situations, there was more variety of processes on predictions than on solutions.

Table 5

Codes And Frequencies For Processes On Prediction And Solution

Code Number	Code Description	Frequency on Predication	Frequency on Solution
0	No Response	17	30
1	Reasoning	14	16
2	Comparison	5	9
3	Visual	15	0
4	Over-Generalization	6	0
5	Other	1	2

Note. Some responses were coded into two categories. Hence, column total is greater than 56.

Relationship to Common Core Standards

When asked to identify CCSSM standards that were well-aligned, some participants identified up to 5 standards with specificity (e.g. 7.G.A.1 or HSG GMD.A.1) while others were much less specific (e.g. 7EE, or 8th grade geometry). The standards range over several grade levels and different content. Those with the larger frequencies are presented in Table 6.

Table 6

Frequency Of Most Common Responses To Standards To Which The Task Aligned

Grade Level and Standards	Frequency
6.RP: Understand ratio concepts and use ratio reasoning to solve problems.	10
6.RP.1: Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.	7
6 G: Solve real-world and mathematical problems involving area, surface area,	8
and volume.	13
7EE: Use variables to represent quantities in a real-world or mathematical problem…	17
7.G.A.1: Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale	28
drawing at a different scale.	3
7.G.B.4: Know the formulas for the area and circumference of a circle and use	3
them to solve problems; …	10
8.G: no specific geometry standard identified	10

8.G.A.4: Understand congruence and similarity using physical models, …	3
HSG C.A.1: Understand and apply theorems about circles.	
HSG GMD.A.1: Explain volume formulas and use them to solve problems.	
HSG GMG: Apply geometric concepts in modeling situations.	

Note. Explanations of some standards are truncated. Abbreviations are: RP-Ratio and Proportion, G-Geometry, EE-Expressions and Equations, HSG-High School Geometry, GMD-Geometric Measurement and Dimension, and GMG-Modeling with Geometry.

Summary

The analysis of the results shows students used a wider variety of thinking and approaches when making predictions than when solving the problem. Similarly, they used more processes in their predictions than in the solutions. This prompts us to continue the research, asking if these differences are a result of past experience, content knowledge, or mathematical viewpoint. The implications for implementing the CCSSM are significant. For instance, effective use of learning trajectories that have been articulated with regard to the CCSSM (Confrey, 2012) may be compromised if teachers do not realize the full potential of tasks suggested in those trajectories. They either may not see how the task embodies the topic at hand or they do not make the connections needed to use ideas from related topics. This also has implications for how teachers use formative assessment practices as well as how they evaluate student work.

It is our contention that while participants acknowledge the connection to the big idea, the lens through which they solved the task affects the way in which they identify the task's alignment to the Common Core content standards, even though they can identify several standards to which the task might align. That is, the demonstrated strategy used to solve a mathematics problem mirrors their view of mathematical alignment of the standards to the task.

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